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THE MECHANICS OF MACHINERY.

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PREFACE.

I FEEL that in venturing to add one more book to the already long list of those which treat of the science of Mechanics, I ought to be able to show that it really does fill some position which has not previously been better occupied. I will therefore offer no apology for summarising here both the scope and limitations of my work.

Most of the following chapters have formed from time to time a portion of courses of lectures on the theory of machines given to my students at University College. They are therefore arranged specially with a view to what I have found to be the wants, requirements, and difficulties of young engineers and students of engineering. Keeping this in mind, and knowing that there is no longer any scarcity of elementary text-books containing a thoroughly sound treatment of general Mechanics, I have confined myself entirely to the mechanics of constrained motion. It is an essential characteristic of every machine that the path of motion of every one of its points is absolutely known at every instant. The *absolute* velocity of any point can be altered, or its motion entirely stopped, but, *relatively to any other point of the machine*, neither the direction of motion nor velocity of any point can be in the slightest degree altered, except by

forces which involve the practical destruction of the whole apparatus. All the motions occurring in machines are thus conditioned by an absolute geometric constrainment which renders it not only possible but very easy to treat them by themselves, and in this fashion to separate the mechanics of machinery from the general science of mechanics of which it forms a portion.

The determination of these constrained relative directions and velocities presents a series of purely geometric problems which are dealt with in Chapters I. to VI. The most certain method for dealing with these problems is, I believe, the method of instantaneous or—as I prefer to call them—*virtual* rotations. I have, therefore, used this method consistently throughout my work, from the very beginning. For the simpler and more numerous problems of plane motion I have taken the virtual centre to replace the virtual axis (p. 41), for conic or spheric motion I have used the axis itself, and for the case of screw motion I have pointed out how the rotation axis must be replaced by the twist axis, of which it is only a special case. In most ordinary mechanisms the virtual axis or centre of every link relatively to every other can be determined very easily for every position. Not unfrequently the centre itself is an inaccessible point, but there are few cases in which this adds any real difficulty to the problem. It is always necessary to know the direction of lines passing through the virtual centre, but it is seldom essential that lines passing through that point should be actually drawn on the paper. In the case of complex mechanisms (such as Figs. 218 to 224 or 244), there is no doubt difficulty in finding the virtual centres. But just in these cases the complete handling of the mechanism by any other means is also a problem of great difficulty, and is sometimes, I believe, almost impossible. The method of virtual rotations, also, lends itself throughout to graphic

treatment, and its difficulties are almost entirely those of geometrical construction, which an engineer who is master of his drawing instruments can easily tackle, and not those of analytical mathematics, with the tools of which he is often, unfortunately, not so familiar. A system which allows every plane and spheric mechanism which has yet found application in machinery, from the simplest cases (Figs. 104 to 109) to the most complex ones (Figs. 128 to 131, 239, 244, 288, 299, etc.) to be treated in exactly the same method and with exactly similar constructions, both in its kinematic and its kinetic relations, possesses on this account advantages so great as quite to outweigh, in my opinion, the very small initial difficulty of thoroughly mastering the idea of virtual rotation which forms its foundation. The theorem of the three virtual centres (p. 73) or axis (p. 490), without which the method of virtual rotations would be practically useless for our purposes, was first given, I believe, by Aronhold, although its previous publication was unknown to me until some years after I had given it in my lectures.

The problems dealt with in Chapters I. to VI. are in reality purely geometric, the velocities dealt with being only the relative velocities of different points in a constrained linkwork. The ideas of acceleration and of force are introduced in Chapter VII. Here I found myself compelled to choose between the adoption of some system of absolute units, and the retaining of the much-abused word "pound" as the name for a unit both of weight and of force. I hope that in § 30 I have succeeded in making clear the vital distinction between these two things, but after the best consideration which I have been able to give to the matter, I have come to the conclusion that the retaining of the word "pound" for both is, for the purposes of this book, the lesser of two evils. Without going further into reasons than I have done in the text, I will only say that the adoption of any other

plan would have made the book practically useless to almost all engineers so long as the thousand-and-one problems of their everyday work come to them in their present form.

In § 28 of Chapter IX. I have given special attention to the construction of diagrams of acceleration from those of velocity, and diagrams of velocity from those of acceleration, showing the constructions necessary in each case both for diagrams on time- and distance-bases. I have found by experience that the only real difficulty in connection with the practical use of these diagrams lies in the determination of their scales. I have therefore gone into this matter in a more detailed fashion than might, at first sight, seem to have been necessary, and have recurred to it frequently in later Chapters.

Problems connected with the static equilibrium of mechanisms are dealt with in Chapter VIII., and with their kinetic equilibrium in Chapter IX. In order to make these latter more complex problems more intelligible to engineers, I have chosen purely technical examples, and worked them out in detail, for the most part graphically. The problems treated include those of trains, "Bull" and Cornish engines, and ordinary steam-engines, while the action of flywheels and of governors is also considered in some detail, and the connecting-rod is used as an example of the kinetic theory of a single constrained link having general plane motion.

In Chapter X. a number of mechanisms intrinsically interesting, but not finding place as examples in the earlier part of the book, are considered. I have here also endeavoured to arrange a general classification of plane mechanisms on a basis which appears to me, at least so far as it goes, to be a scientific one. It has not been consistent with my purpose, or indeed with the size of this book, even to touch upon the enormous number of recently described plane mechanisms (such as many of those of Kempe and

of Burmester), whose interest, although great, is at present entirely kinematic, and the use of which in any actual machine appears extremely unlikely.

The enormous majority of mechanisms with which the engineer has to do have (fortunately for him) only plane motions. To treat non-plane mechanisms with the same detail as plane mechanisms, would have involved a great enlargement of a volume already too bulky. I have in Chapter XI., therefore, not attempted to do much more than to show how the principles already discussed apply to non-plane motion. I have dealt in a detailed manner only with two examples: the universal joint, and Mr. Tower's "spherical" engine. In § 65 I have worked out the action of the latter, kinematically and kinetically, in as complete a fashion as formerly the simpler case of the ordinary engine. The results are shown graphically in Figs. 306 and 307.

In Chapter XII., lastly, I have given some general notion of the influence of friction on the working of machines. In doing this I have put entirely on one side the time-honoured "laws" of dry friction, the relation of which to the friction of machines is purely illusory, and have endeavoured to substitute for them some actual relations between pressure, temperature, and velocity, so far as they are yet experimentally determined, which apply to smooth and more or less completely lubricated surfaces. I would like, however, to emphasise here what I have pointed out in the text (p. 577), that the ordinary calculated determinations of frictional efficiencies have seldom any great absolute numerical value. Not only are the different friction factors very imperfectly known, but the pressures due to "fit," tightening of bolts, etc., which in some cases are more important than any other friction-producing forces, are scarcely known at all.

I have endeavoured to mention throughout the book the names of the various authors and others to whom I have

been, in different matters, indebted. I cannot, however, omit to make special reference to the work of Professor Reuleaux. All engineers are indebted to him for the system of analysis of mechanisms first set forth in his *Kinematics of Machinery*, a system so simple and so obviously true that its essential points have found universal acceptance. The principles of Reuleaux's system I have unhesitatingly made use of, and my first sixty pages are to a great extent a summary of his results. After that our objects have differed widely, and I have followed entirely different lines from those of his work.

I should like to add that although, partly through pressure of work and partly through ill-health, this book appears only now, yet a great part of it has been in type, and a still greater part written and given in lectures, for a number of years. This has rendered it almost impossible for me to make any use of the excellent work of Prof. Cotterill, or of the recently published graphic methods of Prof. R. H. Smith, or the still more recently published (in any complete form) kinematic work of Professor Burmester, as I should otherwise have liked to do.

For a few of my illustrations I am indebted to Reuleaux's *Kinematics*, but nine-tenths of them I have drawn specially for their present purpose. In any cases where engineering details are shown I have endeavoured to draw them with reasonably accurate proportions, and in all cases where diagrams of force, velocity, etc., occur, they will be found drawn to scale, and the scales are marked on the figures.

ALEX. B. W. KENNEDY.

UNIVERSITY COLLEGE, LONDON,
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THE MECHANICS OF MACHINERY.

CHAPTER I.

THE MACHINE.

§ 1. WHAT A MACHINE IS.

As our object in the following pages is to study only a certain section of the science of mechanics, that part of the science, namely, which is involved in questions relating to machinery and mechanical combinations of all kinds, it is right that we should commence with a somewhat precise notion of what the limits of our work are to be. The term *machinery* is a large one; it includes many things differing apparently very greatly from each other, and we must first of all find out what are the common points of all these various machines, the characteristics which belong to them as machines. After this has been done, but not before, we shall be able to see what is the actual nature and extent of the problems which lie before us, what ground we shall have to cover, and what matters we may leave untouched, as relating to conditions which do not exist within the limits of our work.

A machine is a thing which has been very often defined, but very seldom, for some reason or other, with anything

like real exactness or completeness. According to many definitions, either a hammer or a piece of rope is a machine in itself—a bridge is certainly a machine, or a plank resting against a wall. But, in spite of the definitions, we do not as a matter of fact call any one of these things a machine—if we did the only difficulty would be to find anything in the universe that was *not* a machine. It would be useless, in that case, to make any attempt to study the applications of mechanics to machinery as a special branch of the general science of mechanics. It does not, however, appear difficult to include in a single sentence a complete definition of a machine, a definition, that is, which shall include all mechanical combinations which can by possibility receive that name, and rigidly exclude all others. We shall first give a definition which seems to meet these requirements, and then devote the rest of this chapter to its detailed consideration.

A machine may be defined to be a combination of resistant bodies whose relative motions are completely constrained, and by means of which the natural energies at our disposal may be transformed into any special form of work.

In the first place, then, a machine is a *combination of bodies*—a single body cannot constitute a machine. In each of what are often called the "simple machines," for example—the lever, wheel and axle, etc.,—there are at least two bodies, in some more than two. The mere bar which we call a lever does not in itself constitute a machine, either "simple" or otherwise. All its properties depend upon the existence of a fulcrum about which it can turn, and on the position of this fulcrum. Without this the lever is a mere bar incapable of being of the slightest mechanical use to us, with it, (a proper *form* of fulcrum being here sup-

posed, of which more later on) it forms one of the most important combinations with which we have to deal. Here, therefore, a combination of two bodies can be used to form a machine.

Similarly with the "wheel and axle." Wheel and axle themselves form only one body, so far as our work is concerned. That is to say, whether they are originally made in one, or separately, or in a dozen different pieces, they are fixed together rigidly before they are put to any use, the one cannot be moved without moving the other, and we must therefore treat them as one only. But a wheel fixed to a shaft does not of itself make a machine. Before we can utilise it we must provide rigid bearings for the shaft to turn in, and these bearings (themselves also so connected that they form one piece) form a second body in the machine, just as essential to its working as the former. Here, again, then, a combination of two bodies forms a machine;—we shall see later on that this combination is in fact identical with the former.

We might give further illustrations, but these will suffice. There are no cases in existence in which a machine consists of one body only, and indeed we shall see immediately that we are able to say that no such machine is even theoretically possible.

In nineteen cases out of twenty a machine consists entirely of *rigid* bodies. But this rigidity is not an essential condition. Springs of steel or even of india-rubber, which cannot be said to be rigid—which are, in fact, used simply because they are not rigid—often form part of machines. Fluids also are not unfrequently used under suitable conditions. A column of water enclosed in a tube, for instance, may be used to transmit a pressure from one end of the tube to the other, in which case the water itself becomes

in the fullest sense a part of the machine. More frequently than either of these, leather belts or hempen ropes form parts of machines, transmitting motion or force with sufficient accuracy for many practical purposes. Rigidity being thus not necessarily a property of the bodies of which machines consist, the question arises as to what is the essential condition which must be fulfilled by a body in order to make it available as part of a machine. It is that it shall present, or can be made to present, a suitable molecular resistance to change of form or volume, a quality which we have expressed by the use of the word *resistant* instead of *rigid* in our definition. The reason for this necessity for resistance to change of form will be seen better by and by; here, it will suffice merely to illustrate it. We use hempen rope to transmit force (as in tackle) by stretching it over pulleys, and keeping it always in tension. Under these conditions it can transmit pull as well as if it were of iron, and with far greater convenience. If we could so reverse the motion of the apparatus as to put the rope in compression instead of in tension, it would lose its resistance, fail to transmit the pressure, and be entirely useless for our purposes. The case of water is exactly the reverse of this. We can transmit pressure through it, *i.e.*, we can use it in compression, but it is no use as regards pull. We have to enclose the water in a tube, but the tube may be of any form, the fluidity of the water causing it of course to fill up any vessel in which it is placed. The water therefore may change its *form* to a very great extent, but it remains practically unaltered in *volume*, and this resistance to change of volume answers as well for some of our purposes as the resistance offered by a bar of rigid material to the alteration of its length or of its form generally.

The relative motions of the resistant bodies which together constitute a machine are said, in our definition, to be *constrained*. This point requires a little more extended remark than the preceding ones, for it forms in reality the chief characteristic by which problems belonging to the mechanics of machinery can be distinguished from those of general mechanics. It is this, practically, which enables us to separate from the latter, and treat by themselves, all the mechanical questions which arise in engineering work, and specially it enables us (as will be seen further on) to apply to the solution of these questions graphic methods, simple straight-edge-and-compass solutions, which from a practical point of view have in most cases many advantages over algebraic calculations.

The special feature, then, by which the problems belonging to our branch of mechanics are to be distinguished among those of the science in general, is this:—in the general case the bodies whose motions come into consideration are *free*, in all the cases with which we have to do they are *constrained*. Where a body is *free*, the direction in which it moves depends entirely upon the direction of the force which sets it in motion, and can be altered at any moment by an alteration in the direction of that force or, what is the same thing, by the action of other disturbing forces. Where a body is *constrained*, on the other hand, the direction of motion is absolutely independent of the direction of the force which causes motion in the first instance, and can neither be changed by any change in its direction nor by the appearance of disturbing forces. In the first case the motion of a body during any interval of time is only known if we know completely all the forces which will influence it, intentionally or accidentally, during that interval. If a machine had to be treated in this way these forces would

include knocks and jars as innumerable as they are irregular, even a hand pressure on any of its parts would have to be taken into account, and the whole problem would become a totally impossible one. In the second case however, to which the machine fortunately belongs, the direction of motion of every part at every instant is determined completely by its construction. The only possible alternatives are motion in the right direction or no motion at all. As long as the machine is in motion, every part of it is compelled to move in a pre-arranged path, and these paths can only be changed by force if that force be sufficient to distort or destroy the machine. **So long therefore as the machine remains uninjured, we can say that all its motions are, as to their directions, completely independent either of the direction or the magnitude of the external forces which cause them.**

We say that the direction of motion is independent of the direction of the force causing motion. But of course, in machinery as well as in the rest of the universe, the motion of a body is actually determined by the *whole* of the forces acting upon it. In every case we must virtually apply the same method to get rid of disturbing forces—viz.: balance them; but the method of balancing is very different in the two cases of free and constrained motion. In general the resultant of all the forces acting on any body is oblique to the direction in which we wish it to move; such a resultant may be resolved into two components, one in the direction of motion wished, the other at right angles to it. If the latter be balanced, we shall have done all that is necessary to ensure the body moving in the right direction. In the case of a body moving freely this balancing of the disturbing forces must take place from

moment to moment as the latter come into action. If a body be falling to the ground, for instance, and in its motion meet with some disturbing force in the shape of a side push, then in order that its path may not be altered it must be arranged that another push, exactly equal and opposite to the first, shall begin and end precisely at the same instants with it. Any such arrangement for balancing disturbing forces, although theoretically possible, is of course practically out of the question—a totally different plan is used in machinery. *There* we provide for the complete balancing of all disturbing forces by so connecting the moving bodies that any departure from the desired motion could occur only if the *form* of the connections could be changed, and we further make these connections of material such as cannot change its form easily under the forces acting upon it. The disturbing forces, then, are balanced as they occur simply by the resistance of the material of the machine to change of form;—this molecular resistance is usually called **stress**, a word which we shall have frequent occasion to use. We may sum up all this by saying that **the constrained motion characteristic of the bodies which form parts of machines, is obtained by so connecting them that all forces tending to disturb their motion are balanced as they occur by stresses in the bodies themselves.**

It is often said, and to a certain extent quite truly, that the motions of the different parts of a machine are rendered constrained by the geometric *form* (pin and eye, slot, screw, &c.) of the connections between them. When, for instance, we require a body to revolve about a particular axis, we make some portion of it in the form of a cylinder having the same axis (Fig. 1), and cause this cylinder to work in bearings made to fit it accurately—the cylinder being

provided also with projecting rings or collars fitting the sides of the bearings in order to render endlong motion impossible. It is true that in this and in every other such case the ***kind of motion*** permitted is determined by the

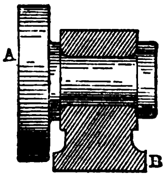

FIG. 1.

form of the connections. But it is not sufficient merely to give the bearings the correct form in order to prevent undesired motions. In the case supposed, for instance, bearings of india-rubber, however accurately made as to their shape in the first instance, would quite fail to constrain the motion—their resistance to change of form would be insufficient to balance the disturbing forces occurring during the work of the machine. To obtain constrained motion, therefore, it is necessary to do more than merely employ proper forms in connecting the different parts of the machine: the connections must also be of proper *material*, and the constrainment must ultimately be referred to the molecular resistance of the material itself rather than to its form.

We cannot find any materials which shall resist *all* disturbing forces; if we could do so we could make an unbreakable machine. But we can readily obtain materials such as shall remain without any sensible change of form under the action of all the forces occurring in the ordinary working if the machine to which they belong. Using

such materials, and the forms of connection suited to the particular motion wanted, we obtain a completely constrained motion. Recurring to the illustration used in the last paragraph: unless the form of the bearings can be changed, the shaft can only rotate about its own axis; we make the shaft of iron, and bearings of brass and their size such that their resistance to change of form is greater than any external force which in ordinary working can be present to change their form. This remains therefore unchanged, and with it the motion also remains unaltered.

It is of course possible for a machine to be injured, or even broken, by the occurrence of too great stress in some of its parts. But it is the province of the designer of the machine to provide against these contingencies, and—as they can always be avoided—it is not necessary for us to take them into account here. For our present purposes therefore we shall always assume that the machines or mechanisms with which we are dealing are so designed as to give us complete constrainment of motion in all their parts. The direction of the motion will be determined in every instance by the form of the parts, and all forces or components of forces tending to disturb such motion will be assumed to be balanced by the stresses in those parts; the nature and magnitude of the stresses as well as the minute alterations of form which always accompany them, forming subjects which we shall have to examine later on.

The possibility of making this assumption very greatly simplifies the treatment of our subject, as can easily be seen. We can consider the motions occurring in a machine quite independently of the forces acting upon it. The paths in which the different points move, as well as all their *relative* velocities, can be determined by purely geometric methods, without touching static questions at all.

The consideration of force comes in only when we have to deal with the equilibrium of the machine, or the *absolute* velocity of any of its parts. Even then there are many problems in which we may neglect all force-components but those in certain (known) directions, knowing that the others will necessarily be balanced by stresses which are equal to them in magnitude, but the knowledge of whose magnitudes is not required in any way for the purposes in hand.

We have now left only the last clause of our definition to examine, which defines the object of a machine as the transformation of natural energies into some special form of work. A machine is often spoken of as "an instrument for transmitting and modifying *force*." It is as well to remember that although a machine does transmit and modify force, yet this is by no means a special characteristic of the machine. A bridge, a roof, everything in fact which we include under the name *structure*, does the same. No machine was ever constructed merely to modify force; *motion* is essential to the machine, and force in motion is work or energy. The special work to be done may be lifting a weight, shaping a piece of metal, spinning a thread, or any one of a thousand other things. The energy which we transform into this particular work may be muscular energy, gravitation energy, electrical energy, or—as in the great majority of cases—heat energy. In every case, however, the object of the machine is to utilise one or other of these forms of energy by transforming it into some particular kind of mechanical work which we require to be done. It is precisely this which distinguishes a machine from a mere structure—the latter modifies force only, not energy—as when the weight of a train is resolved into the upward forces which support it at the abutments of a

bridge, appearing between the one and the other in numerous modified forms as stress in the various bars and beams of which the bridge consists.

We have defined and discussed what may be called a *perfect* machine. In practice we meet with a good many combinations (especially those in which non-rigid bodies are used), which are essentially machines, but in which the motion of the different parts is very far from being *completely* constrained. A pulley tackle hoisting a weight is a familiar illustration of this—the motion of the weight being what we have defined as *free* in every direction but the vertical. But in just so far as free motions occur does the whole apparatus depart from essentially machinal conditions, and in just so far must it be regarded as, from our point of view, imperfect or incomplete. Again there are some familiar combinations, such as stop valves, safety valves, etc., which often form parts of more important apparatus, and which in any case are as fully fulfilling their ultimate object when their different parts are stationary as when they are in motion. Of these it may be said that whether or not they are to be called machines—which is matter of indifference—they yet behave as machines during the time when they are in motion. They may therefore be handled by just such methods as we find applicable to machines, with the qualification only as to the completeness of their constraint which has just been mentioned.

Many problems, of course, occur in connection with the machine, in which it is possible to leave motion entirely out of consideration—to treat the machine in fact merely as a structure. To this class belong all the ordinary **static** problems relating to the equilibrium of mechanical combinations, with which we shall have to deal shortly. There is also another class of questions not affecting the

transformation of energy, or even of force, but dealing simply with the relative motions of the different parts of the machine. Such problems as these are called **kinematic,** and on account of their great simplicity we shall take them up before touching any of the others. Following these we shall come to the **statics of machinery,** and lastly to the consideration of the more complex questions which arise when we have to consider the machine in its complete function—the **kinetics of machinery.**

§ 2. THE PRINCIPAL FORMS OF CONSTRAINED MOTION.

THERE is no impossibility in constraining any kind of motion whatever in a machine, however complex that motion may be. An ordinary sewing machine, for instance, affords a familiar illustration of the constrainment of motion of no small degree of complexity. But the immense majority of motions actually utilised, and among them all the more important ones, fall under certain special cases which makes their treatment comparatively simple. We must notice briefly the nature of these special cases, the three principal of which we may call plane motion, spheric motion, and screw motion or twist, respectively.

Plane Motion. When a body moves so that any one section of it continues always in its own plane (Fig. 2), then every other plane section parallel to the first moves also in its own plane, and the motion of the body is said to be "con-plane," "complanar," or we may call it simply, "plane." The enormous majority of the motions occurring in machinery belong to this class; every single motion, for instance, in an ordinary steam engine, with the exception of

that of the governor balls when they are rising or falling is plane. In considering such motions, as well as the innumerable problems connected with them, we can use the very important simplification that instead of dealing with solid bodies as such, we may treat each body as if it

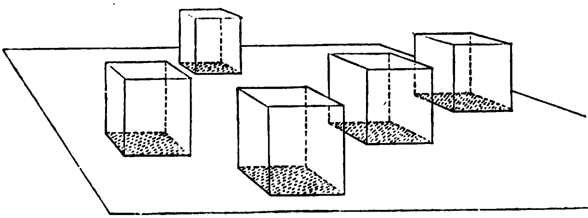

FIG. 2.

were merely a section of itself, *i.e.* a *plane figure.* Such a figure moves in its own plane, and therefore its motions can be completely and exactly represented or copied on the plane of our paper, without the aid of perspective or other projection. As the body, moreover, may be looked upon as consisting of a series of such figures or sections all parallel to each other, and all having exactly similar motions in parallel planes, the motion of the one figure represents that of all the others, that is, of the whole body. In such a case it is indifferent whether we speak of the motion of the figure only, or of the body, the one determines the other; we may sometimes use the one and sometimes the other expression, as may be most convenient.

About the general characteristics of plane motion we shall have a good deal to say further on, at present we may notice in passing that there are two special forms of such motion of particular importance to us. The first is when the motion is a simple rotation. When a *body* rotates about an *axis*, every plane section of it at right angles to that

axis moves always in its own plane, and rotates about the *point* which is the intersection (or to employ a very useful and much needed contraction, the *join*) of the axis with its plane. It is in general a matter of mere convenience whether we treat this motion as a turning about an *axis* or about a *point* or *centre*; in the one case we refer to the body itself, in the other to the plane section or figure which represents it. In any case the path of every point in the body is a circle about the given point or axis, all points at the same distance from which describe equal circles.

If we suppose one point to be turning about another point in this way, and the centre of rotation to be moved farther and farther off, the circle described by the moving point becomes flatter and flatter; any arc of it, that is to say, more and more nearly approaches to a straight line. So long as the centre is at a finite distance however, no matter how great, this line still remains really an arc of a circle, however closely it may be made to approximate to a straight line. But as we find that this approximation grows more and more close the further the centre is removed, it has become a common custom among mathematicians to say that if the centre were removed to an *infinite* distance the circular arc would actually become a straight line. From this point of view a straight line is a circular arc of infinite radius, or one whose centre is at infinity. What there is at infinity we naturally do not know, but we know that we may make this assumption and apply exactly the same reasoning to it which we apply in connection with circles of finite radius, and use all the same graphic constructions also, without coming to any results which contradict the assumption. On the other hand, as will be seen more fully further on, this assumption is one which often leads to very important simplifications, and allows

of very easy solutions to what would otherwise be exceedingly complicated problems. We shall therefore have very frequent occasion to use it, the more because of the intimate connection it has with the second most important special case of plane motion. In this case all points in the body move in parallel straight lines, and the whole body therefore moves "parallel to itself." This is the simplest case of what is called a motion of *translation*, and we may obviously define it, in accordance with what has just been said, to be a motion of rotation about a point at an infinite distance. There is no geometric difference between rotation and translation, and by treating the latter as a special case of the former—as being, namely, a rotation about an infinitely distant, but nevertheless quite easily determined point—we can in many cases avoid the double constructions and double proofs which otherwise would be necessary. We shall find, indeed, that the constructions used in connection with points at infinity are generally simpler and easier than those employed for points at a finite distance.

Spheric Motion.—When a body moves so that every point in it remains always at the same distance from some fixed point, it is said to have spheric motion. If we take a section of the body cut by a sphere having the given point as centre (Fig. 3), we get a figure whose motion is such that it remains always on the surface of a sphere of its own radius. The condition is exactly analogous to that of plane motion, with the substitution of the sphere for the plane, and the spheric section for the plane section. Just as before, the motion of one section of the body—now a spheric section—represents for us the motions of all the others, *i.e.* of the whole body. If we suppose the sphere to take a radius larger and larger until it becomes infinite, we get

a motion more and more nearly resembling plane motion until at length it coincides with it. It would therefore be both possible and scientifically correct to consider plane

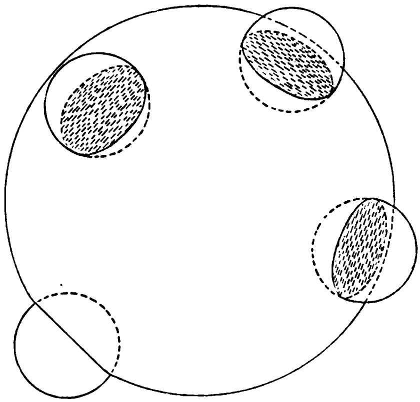

FIG. 3.

motion simply as a special case of spheric motion, but the small simplification which might thus be obtained is not sufficiently important to justify a change which would have some much more considerable practical inconveniences.

Twist.—When the motion of a point is such that it can be resolved into a rotation about an axis, and a translation parallel to the axis, so that the amount of the one is always proportional to that of the other, the point is said to move in or describe a helix or regular screw line.[1] It is possible for a body to move so that all its points describe helices about an axis, and such a motion is called a twist, or screw motion (Fig. 4). Each point in the body remains at a constant

[1] Of *general* screw motion, so interesting to the mathematician, nothing requires to be said here, for reasons which are sufficiently obvious.

distance from the axis, and every point moves through the same angle of rotation and through the same distance parallel to the axis in the same time. The amount of translation corresponding to one complete rotation about the axis is called a *pitch* of the helix :—the helices described by the different points of the body are all equal in pitch

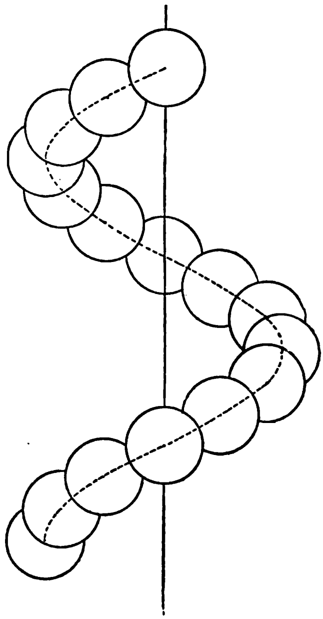

Fig. 4.

but vary in diameter. All points at the same distance from the axis describe congruent[1] helices. Twist is a

[1] The word *congruent*, which may be unfamiliar to some, means *similar* and *equal*.

motion tolerably often met with in machinery, although seldom used for its own sake. It stands in a very simple relation to plane motion, into which it resolves itself in two limiting cases;—viz.: when the pitch of the twist is reduced to zero, and when it is increased to infinity. In the former case the twist becomes a mere rotation, in the latter a mere translation, and these motions therefore might be considered as special cases of twist if there were any object in doing so. The two simpler motions are, however, of so great importance for their own sake that we shall find it advisable rather to treat them in the way we have already indicated, as cases of plane motion, than as special cases of twist.

In the foregoing paragraphs we have pointed out some general characteristics of the principal forms of motion which occur, constrained, in mechanical combinations. More general and complex motions do occur occasionally, as in the case of the sewing machine already cited, and as in some agricultural machinery, but comparatively very seldom; while of the motions mentioned, the very simplest —plane motion—is incomparably the most important and the most often met with. Without, therefore, confining our attention solely to plane motions we shall have very much more to say about these than about any others. The problems arising out of or in connection with more general motions are in very many cases both too complex and too technical for treatment here. Some of them we shall, however, look at, and while we shall be able to treat fully but a few, we shall endeavour so to indicate the methods by which they can be handled, that students who wish to follow up this part of the subject may not find any difficulty in doing so.

In chapters II. to X., however, we shall concern ourselves

exclusively with plane motion, and the statements made and constructions given in those chapters must all be taken with this limitation, unless it is expressly stated that they refer to some more general form of motion. Attention is here drawn to this limitation once for all, to avoid the necessity of frequent qualifying references to it in what follows. The propositions in § 3 of this chapter are, however, quite general, applying equally to the most complex and to the simplest motions.

§ 3. RELATIVE MOTION.

We know that all bodies around us, whether they appear fixed or moving, are continually changing their position in space, but we are unable either to realise or to measure these changes of position, which constitute what is called their *absolute motion*. When a body appears to us to be in motion, what we observe is that the distances between certain points in that body and certain points in some other body undergo alteration, and this we express by saying that the first changes its position—or, in one word, moves—*relatively* to the second. The choice of this second body, the standard by which the motion is observed, is arbitrary. In general it itself has no visible motion relative to any other body. In the majority of cases, for instance, we speak of a body as moving or stationary according as it is changing or not changing its position relatively to the earth, the motion of which, for any short period, is not perceptible to our senses. Often, however, we take some body which is itself moving relatively to the earth, such as a train or a ship, for a standard, and call those bodies fixed which are not moving relatively to it, no matter what motion they may have relatively to the earth. It is quite

easy to suppose a case in which a body in a train is moving in a direction exactly opposite to that of the train's motion and with an equal velocity. Such a body would have no motion relatively to the standard by which the motion of the train itself was observed, *i.e.* relatively to the earth. It would therefore be called stationary if the earth were the standard, although it is moving relatively to the train. A person seated in the train, on the other hand, although moving relatively to the earth, would be said to be stationary relatively to the train.

It becomes, therefore, important that we should form some exact idea of what is implied by the word stationary, of what is the condition, namely, common to the two cases just mentioned. It is a very simple one. In the first case supposed, the stationary body shared the motion of the earth, in the second, it shared the motion of the train; in both cases, that is, it shared the motion of the standard relatively to which motion was measured. If a body be stationary relatively to any other, it shares all the motion of that other; and when we say simply that a body is fixed or stationary, we assume tacitly that it shares all the motion of the standard relatively to which change of position is measured—its (unknown) absolute motion must be the same as that of the standard. The result is the same as if the standard were itself absolutely at rest.

It is obvious that if a body be at rest relatively to another it may be considered as virtually forming a part of that other. The two might be rigidly connected and made one without any change in the conditions. We shall find that it is often convenient to treat a stationary body as simply a part of the body which is the standard relatively to which motion is measured.

In mechanism and machinery change of position is very

generally measured relatively to the frame of the machine, and this is most often stationary relatively to the earth. It is frequently necessary, however, to examine the relative motions of two portions of a machine both of which are moving relatively to its frame. It is of such great importance to get the idea of relative motion under these different conditions clearly realised, and we shall have to use it so frequently, that it will be worth while to examine it a little more in detail.

We have seen that when a body has the same motion as the standard it is said to be at rest. Two bodies, therefore, which have the same motion as the standard, must be at rest relatively to each other, *i.e.* they can have no relative motion. The converse proposition however—that if two bodies have no relative motion they must have the same motion as the standard—is not necessarily true, but expresses only a possible condition. For we have seen that the choice of the standard is quite arbitrary; in the case supposed therefore, the standard may have an infinite variety of motions, only one of which can be the same as that of the two bodies. At the same time if the bodies have no motion relatively to each other, no alteration in the standard can give them any. Whatever the standard chosen, however, the two bodies will have *the same motion relative to it.* This will be easily recognised when it is remembered that, as has been pointed out, two bodies having no relative motion form to all intents and purposes parts of one rigid body; they therefore cannot have different motions relatively to any other.

We are now able to state in general terms the proposition:—**If two bodies have no relative motion they must have the same motion relatively to every other body.** This carries also its converse with

it:—**If two bodies have the same motion relatively to any other, they have no relative motion.** Besides these propositions we have also the important corollary that **the relative motion of two bodies is not affected by any motions which they may have in common.** For whatever the common motion may be, whether absolute or relative, it leaves the bodies relatively at rest, and therefore cannot alter their relative position.

Illustrations of these propositions are very familiar to us. Bodies on the surface of the earth have no motion relatively to each other; we call them all stationary, for they share the absolute motion of the earth as well as its motion relatively to the sun or any other standard. The relative motions of the different parts of a marine engine are not affected by the complex motion of the ship relatively to the water, for all parts have these in common; and so on.

We have had occasion to speak several times of two bodies having "the same motion." The idea of different bodies having the same change of position is not, perhaps, so simple as it appears; it will be well to look more closely at it. One body is said to have the same motion as another when the two bodies could be rigidly connected together during the motion without any alteration in it. We have already seen that this is a consequence of the one body having no motion relatively to the other, or of both bodies having the same motion relatively to any third. But it is necessary that we should look into this matter somewhat more closely than this. It is a well known theorem that a body may be moved from any position to any other whatever by giving it two motions—a motion of translation through a certain distance and a motion of rotation about a certain axis, and that this may be done in an infinite number of different ways. Every motion, therefore, con-

sidered as a change of position of finite extent, may be divided into the two simple motions—translation and rotation. But in each of these cases separately the meaning of equal motions can easily be understood. If a body have a motion of translation through any distance, a second body will be said to have the *same* motion if it be translated in a *direction parallel to the first, in the same sense,*[1] *and through an equal distance.* Similarly if a body have a motion of rotation about any axis through any angle, a second body will have the *same* motion if it be turned in the *same sense through the same angle and about the same axis.* As every motion can be decomposed into a translation and a rotation, we may say therefore that those motions are the same which are composed of equal translations and equal turnings.

Fig. 5 may make this somewhat more clear. *AB* and *MN* are contemporaneous positions of two bodies; the

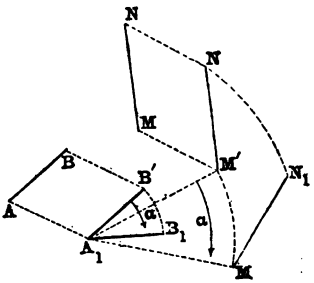

FIG. 5.

former moves (that is, changes its position) to A_1B_1,—let it be required to give to *MN* the *same* motion, or change of position. The motion of *AB* may be resolved into a

[1] See p. 38.

translation to A_1B' and a rotation through the angle α about A_1.[1] By making MM' and NN' parallel and equal to AA_1 or BB' we find the position $M'N'$ which MN would occupy after a translation equal to that passed through by AB. Further by making the angles $M'A_1M_1$ and NA_1N_1 each equal to α, $A_1M_1 = A_1M'$ and $A_1N = A_1N'$; we find the position M_1N_1 which MN will occupy after a rotation about A_1 equal to that of AB about the same point. In order to have a motion, therefore, the same as that of AB in moving to A_1B_1, MN must move to M_1N_1. Fig. 5 serves also to illustrate the proposition that two bodies having the same motion relative to another have no relative motion. The standard is in this case taken as the plane of the paper. The relative positions of A_1B_1 and M_1N_1 are the same as those of AB and MN. If MN had been rigidly connected to A_1B_1, and AB had been moved to A_1B_1, MN would have taken the position M_1N_1, which we have already found for it.

We have seen that the relative motion of two bodies is not affected by any motions which they may have in common. In studying the relative motion of bodies we may therefore neglect all such common motions, a procedure which greatly simplifies many problems. But we may go further than this, we may not only subtract, but may *add* common motions, and this is often extremely convenient. It is specially useful in problems involving the relative motions of two bodies both of which are themselves moving relatively to the standard. Such problems can be at once simplified by supposing added to the motion of both bodies a motion equal but *opposite* to the motion (relatively

[1] In the figure the motions are indicated (for simplicity's sake) as if they occurred in, or parallel to, the plane of the paper. The point A_1 about which turning takes place, must be regarded as the projection on that plane of an axis which is perpendicular to it.

to the standard) of either one of them. The one body has therefore no change of position, that is, it is "brought to rest," relatively to the standard, and the whole motion of the other body relatively to the same standard becomes its motion relatively to the first body. Relative motions which are otherwise very difficult to realise, can in this way be made to appear quite simple and easy of comprehension.

An illustration may make this more clear. Let it be required to find the relative motions of the bodies AB and MN (Fig. 6), during the motions AB . .. A_1B_1, MN . . .

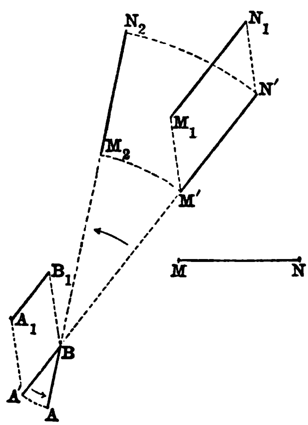

FIG. 6.

M_1N_1. As motions common to the two bodies do not alter their relative positions we may give to both first a translation through a distance $=B_1B$, in the sense from B_1 to B—which brings them into the positions $A'B$ and $M'N'$ respectively—and then a rotation through an angle $A'BA$ about the point B. At the end of this rotation MN occupies the position M_2N_2, while AB has returned to its original position. The

body AB has therefore made no motion relatively to the paper, and the change of position MN . . . M_2N_2 is the motion of MN relatively to AB, which we required to find.

We have supposed here that we gave to both bodies a motion equal and opposite to that of AB. We have thus brought that body to rest and can at once see the whole motion of MN relatively to it. We might equally well have given to both bodies a motion equal and opposite to that of MN (Fig. 7). We should then have brought MN to rest,

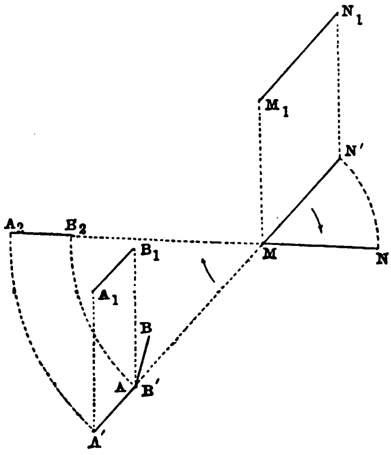

FIG. 7.

and made the whole motion of AB relatively to it visible. But the relative motions have remained unchanged throughout by hypothesis. The motion therefore of MN relatively to AB when AB is fixed, is the same as that of AB to MN when MN is fixed. We may sum this matter up in a general

proposition, for which we shall find frequent use. **If A and B be two bodies moving relatively to each other, the motion of A relatively to B is the same as the motion of B relatively to A, and is the same whether both bodies be moving or either one stationary relatively to any particular standard.**

Here, however, the sameness of the motion does not include *sense*, but merely magnitude and direction. It will be remembered that we are not here limiting ourselves to plane motion, but that the actual translations may be in any direction in space, and the actual rotations about any axis parallel to that direction. Such translation and rotation together constitute some form of *screw motion*. If this screw motion of A relatively to B be right-handed, that of B relatively to A will be left-handed, and *vice versâ*. In Figs. 6 and 7 it is only the sense, or "hand," of the rotation which is seen to be altered, the absence of perspective not allowing the screw motion to be seen.

CHAPTER II.

PLANE MOTION.

§ 4. RELATIVE POSITION IN A PLANE.

We have defined motion, so far as we are now studying it, as change of position. We have seen also that we have to consider only the change in the position of one body relatively to another, and not the absolute motions of bodies. We shall now commence the more detailed treatment of this branch of our subject.

It is necessary first to examine the general conditions by which *relative position* is or may be determined. Just as the absolute *motion* of a body in space is a matter which does not concern us, so the absolute *position* of a body in space or of a figure in a plane is indifferent to us. We can assume a point or a figure stationary in any part of the plane, our object is solely to examine the position of others relatively to it.

Starting then with the notion of a fixed point in a plane, we have first the proposition that **the position of one point relatively to another is determined solely by the distance between them.** It is entirely unaffected by the *position* of the line joining them. Thus in Fig. 8, the points A and A_1, which are at the same distance from P, have the same position relatively to it, and, generally, all points in

a circle occupy the same position relatively to its centre for the same reason. A point having no angular magnitude, that is, no *sides*, there cannot be any differences of angular position relatively to it. It is evident, however, that the points AA_1

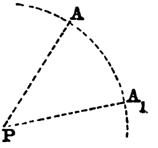

FIG. 8.

&c., occupy different positions in or relatively to the plane in which they are. We see therefore that **the position of a point in a plane is not determined by its position relatively to a point in that plane.**

A *line* is fully determined if two of its points be known. **The position of a line relatively to a point is therefore known if the positions of two of its points relatively to the fixed point be known.** These

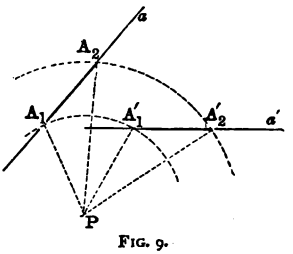

FIG. 9.

positions are determined, as mentioned in the last paragraph, solely by distances from the fixed point. As long as these distances are the same the position of the line relatively to the point is the same also. Thus in Fig. 9, where $A_1P = A'_1P$

and $A_2P = A'_2P$, the position of the line A_1A_2 relatively to the point P is the same as that of $A'_1A'_2$ relatively to the same point. But these lines are in different positions in the plane—hence **the position of a line relatively to a plane is not determined by its position relatively to a point in the plane.**[1]

The position of a *point* relatively to a *line* may be determined in two ways. It is known (i) **if its distances from two points of the line be known, (ii) if the positions of the lines joining it to two points of the line be known.** Thus in Fig. 10 the position of the point A

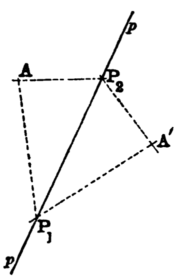

FIG. 10.

relatively to the line P_1P_2 is determined by (i), if the distances AP_1 and AP_2 be known, or by (ii) if the angles AP_1P_2 and AP_2P_1, made by AP_1 and AP_2 at the points P and P_2 of the line, be known. But we can always find two points in the plane, one on each side of the line, which shall satisfy any given conditions either in (i) or (ii). The point A', for instance occupies the same position relatively to P_1P_2 as A. A point may therefore occupy two

[1] It may be noticed in passing that the theorems just given are equally true whether or not all the points or lines are in the same plane. They hold good, that is, for spheric equally with plane motions.

positions in the plane for all positions which it can take relatively to any line in the plane, so that its position in the plane is not absolutely determined by its position relatively to a line in the plane.

We can, however, adopt some simple convention to distinguish between the two parts into which the line divides the plane; taking distances measured from P_1P_2 as positive to the one side and negative to the other, for instance. If we suppose this to be done, the symmetrical positions A and A' can be distinguished from each other, and the position of A in the plane is by this means determined when its position relatively to the line P_1P_2 is known.

The position of one line relatively to another in the same plane is known if the positions of two points in the first are known relatively to two points in the second. Here again we have an indeterminateness of the same kind as in the last case. A line may occupy two different positions in the plane, as A_1A_2 or $A'_1A'_2$ Fig. 11, and yet be in the same

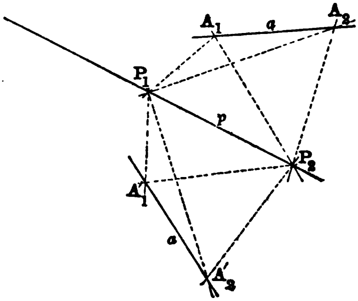

FIG. 11.

position relatively to a line P_1P_2 in the plane. If these positions be distinguished by such a convention as that just alluded to, however, the indeterminateness disappears,

and we may say that **the position of a line in a plane is determined by its position relatively to any other line in the plane.**

If *a*, Fig. 12, be any given plane figure, and *A B* any two points in that figure, then if we know the positions of

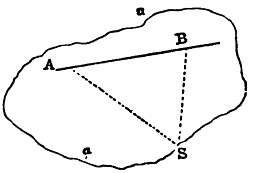

FIG. 12.

these two points we know the positions of all the others. For any other point, as *S*, can be found at once as the vertex of a triangle of which the magnitudes of all three sides (as *SA*, *AB*, *BS*) are known. **The position of a plane figure in a plane is therefore known if the positions of two points—that is, of a line—in it be known relatively to two points in the plane.**

If we discard the convention of positive and negative alluded to above, the position of a point in, *i.e.* relatively to, a plane is known only if its position relative to *three* other points in the plane, not in the same straight line, be known. Similarly the position of a line, and consequently of a plane figure, in a plane, is only completely determined if the positions of two of its points relatively to three points in the plane—not in the same straight line, be known. For our purposes, however, the two points will generally be sufficient, it is seldom that the circumstances of the case leave any doubt as to which of the two possible positions is the required one.

§ 5. RELATIVE MOTION IN A PLANE.

We have seen in the last section the conditions necessary to determine the relative positions of points, lines and figures in a plane. The *motion* of a point or line, however, is represented to us by the series of different *positions* which it occupies relatively to another point or line, &c. Each one of these is determined by the same conditions, so that the conditions which determine the *position* of the point or line relatively to any other, determine also its *motion* relatively to that other. We get therefore,—in most cases by little more than verbal alteration,—the following propositions as to relative motion in a plane, corresponding to those of the last section as to relative position.

One point can move relatively to another only along the line joining them. Thus in Fig. 13, A

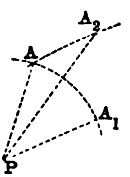

FIG. 13.

does not move relatively to P in moving to A_1, because every point in AA_1, its path of motion, has the same position relatively to P. In moving from A to A_2, however, A moves through the distance $PA_2—PA$ relatively to P.

The motion of a line relatively to a point is determined by the motion of two points in it relatively to that point. Each of these points can move, relatively to the fixed point, only along the line

joining them. We see then that (just as in the case of position) the motion of a point or a line relatively to a plane is not determined by its motion relatively to a point in that plane. If a line turn about a point, for example, it remains stationary relatively to that point, although it is in continuous motion relatively to the plane.

The motion of a point relatively to a line is determined by its motion relatively to two points of the line.

The motion of a line relatively to a plane in which it moves (or to a line in that plane), is determined by the motions of two points in the one relatively to two points in the other.

And lastly **the motion of any plane figure relatively to its plane is determined by the motions of any two points, *i.e.* of a line, in it**

The last theorem may be stated also in another way. The figure being supposed rigid, no point in it can move relatively to any other,—all points in it, therefore, must have the same motion. But this motion is that of any line in it. When we have given, then, the motions of any two points whatever in a figure, we know the motion of the figure, and we know also that the motion of every other point in the figure is the *same* (in the sense already explained) as the known motions of the two arbitrary points with which we started.

We have already seen that when a body has plane motion the whole motion of the body is known when that of any plane section of it, moving in its own plane, is known :—the motion of the section or figure represents that of the whole body. But we have now seen further that the (plane) motion of a figure is known if the motions of two of its points be known. **The plane motion of a body,**

therefore, is known if the motion of any two points, that is of a line, in any of its sections parallel to the plane of motion, be known, and all the theorems just enunciated as to the determination of the motion of a line apply equally and absolutely to the determination of the plane motion of the body to which that line belongs. Thus for instance, the motion of the whole body shown in Fig. 2, is determined by that of any such plane section of it as the one shaded in the figure, and the motion of that section again is determined by the motion of any two points in it.

§ 6. DIRECTION OF MOTION.

We have been considering motion as a sequence of changes of position, each of finite extent. Each such change occupies some finite interval of time, at the beginning and end of which the body occupies different positions. Instead, however, of considering completed changes of position in this way, it is often necessary for us to examine the change of position which a body is actually undergoing *at some particular instant.* This is called the **instantaneous motion** of the body.

As the body moves every point in it describes some curve in the plane, and it is sometimes convenient to use the name *point-paths* for such curves. To know the whole motion of the body we must know these point-paths, or as many of them as give us the means of knowing all the rest; to know its instantaneous motion we require only to know the *direction* of the point-paths at the given instant. By the direction of the point-path at any instant is meant the direction in which the point which describes that path is

moving at that instant, that is, the direction of a line joining the point with the next consecutive point of the curve it is describing, which is, of course, infinitely near to the first. But a line which joins two consecutive points of a curve is called a tangent to the curve. Two such points cannot be any finite distance apart, or it would be possible to find another point between them, and they would not be consecutive. We therefore assume the distance between them to be infinitely small, or in other words we assume them to coincide. A tangent therefore,—a line joining two consecutive points of the curve,—is by definition a line passing through two points of the curve, but it differs from all other lines which have the same property in that these points are coincident. Fig. 14 may make this clearer. Suppose the

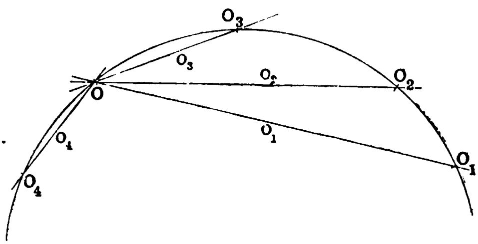

FIG. 14.

line o to turn from o_1 to o_2, o_3, etc., about the point O in the curve. It cuts the curve always in O and in some other point, and this point moves continuously along the curve, taking the positions, $O_1 O_2 O_3$ . . . . O_4 &c. In doing so the second point must have passed through O itself, for it has passed over from one side of O to the other. When the line occupies this central position, its two points of intersection with the curve are said to coincide at O, and it is called the tangent to the curve at O.

When a point then is moving in any curve, its direction of motion at any instant coincides with the direction of the tangent to the curve drawn through the point.

By reasoning similar to that adopted in the last two sections we have then at once the following propositions relating to instantaneous motion. **The instantaneous motion of a point is known if its direction of motion, *i.e.*, the tangent to its path, be known for the given instant.** Here again we have conditions similar to those which were examined in §§ 4 and 5; the path of the point relatively to another point is not, in general, the same as its path relatively to the plane. Its instantaneous motion will differ, therefore, according to the standard relatively to which it is observed, just as its change of position does.

The instantaneous motion of a line is known if the directions of motion of (or tangents to the paths of) any two of its points be known for the given instant. In both cases it is only point-paths or directions relatively to the plane with which we need concern ourselves at present.

We have seen that the motion of any plane figure in the plane can be fully determined from the motion of any two of its points. This is as true in the case of instantaneous motion as in the case of finite change of position. The former differs from the latter only in that the changes of position are regarded in it as being indefinitely small. It requires two points (assumed to be indefinitely near together) to determine each tangent, and these are simply the two consecutive positions of one point when its change of position has become infinitely small. We get, therefore, the important proposition that **the instantaneous motion of**

a plane figure in its plane is determined by that of any two of its points. And from this follows the very important corollary that **the directions of motion of all points in a figure are fixed when those of two points in it are fixed.**

The direction of motion of a point, in the sense in which we have been using the word, is given us by a line. But a point may be said to move in either of two "directions" along this line. To avoid any indistinctness from this double use of one word we shall restrict "direction" to the former meaning, using "sense" for the latter. A line, then, determines a *direction*, while along that line a point may move in either of two *senses*, which we shall often have to distinguish. In writing of them we may call one positive, and the other negative; in figures the particular sense of motion can be shown always by an arrow. The word "sense" is used similarly in reference to a line turning about a point. It may be turning either clock-hand-wise, or in the opposite *sense*.

§ 7. THE INSTANTANEOUS OR VIRTUAL CENTRE.

The instantaneous motion of any point is completely known, as we have now seen, when the *direction* of its motion is known,—it is therefore quite independent of the form of the curve in which the point is moving. A point, therefore, of which the position and the direction of motion are known may have for its actual path any one of the infinite number of different curves which can be drawn touching the given direction line in the given point. For example, in Fig. 15 a point O is moving at a particular

instant in the direction of the line p. But its actual point-path may be o_1, o_2, o_3, o_4, or any curve whatever which is touched by the line p at the given point. This fact enables us to simplify our problems enormously, so far as they have

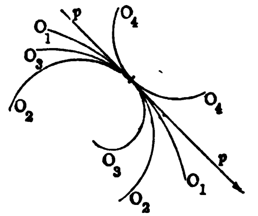

FIG. 15.

to do with instantaneous motion. In all machines certain leading and very important points have very simple motions,—rotation or translation,—but in all machines except the very simplest there occur constrained motions by no means so simple as these, and not unfrequently really complex. Sometimes we are concerned directly with the form of the motion in such cases, and then it has simply to be worked out in the most direct way possible. More frequently, however, as will be found, we are concerned only with the instantaneous motion of each body forming the machine, and with the directions in which its points are moving at some particular instant. In this case it becomes very easy to substitute for the actual complex motion an imaginary simple one which for the instant is identical with it, and which admits of treatment of the most direct possible kind. This we can do in the following way :—Let a be any plane figure (Fig. 16), and A and B any two points in it, of which the directions of motion, a and b respectively, relatively to the plane β, are known. By these data the instantaneous motion of the whole

body is, as we have seen, determined. Let AO and BO be perpendiculars drawn to a and b at the points A and B. Then about every point in AO we can draw a circle touching a in A, and the instantaneous motion of A, whatever its

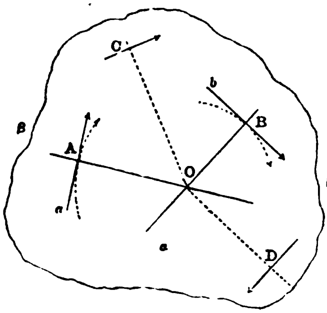

FIG. 16.

actual path, is the same as if it were moving in any one of these circles. Similarly we can draw a circle about every point in BO touching b in B, and any one of these circles if it were the path of B would give it the same instantaneous motion as that which it actually has. But AO and BO, being lines in the same plane, must have one point in common, their intersection or join,—here the point O. If, therefore, A and B were both moving round this point as a centre their instantaneous motion would remain just what it is,—for the lines a and b would still be the direction of motion of A and B, or tangents to their paths; but these paths would now be circles having O as their common centre. It has already been shown that the plane motion of any figure is the same as that of any line in it. In this case the line AB is for the instant simply rotating about the point O,—**the motion of the whole figure α, therefore, is simply a rotation about the point O, which**

is called its instantaneous centre. We shall find, in what follows, that this point is of continual importance to us, and that it has to be very frequently spoken of. It is therefore desirable to have a somewhat shorter and less unwieldy name for it than that just mentioned, the one commonly used. We propose, therefore, to call such a point a **virtual** rather than an *instantaneous* centre. It is a point about which a figure or body is virtually moving at the instant at which its motion is under consideration. The plane motion of any *body* represented by the figure a may similarly be treated as a rotation about an *axis* perpendicular to the plane of the paper, and of which the point O is the trace in that plane;—such an axis is called the *instantaneous* or **virtual axis** for the motion of the body. It is important to remember that the centre O is in reality only the projection of such an axis, but for reasons which we have already given it is generally more convenient to speak of the *figure* than of the *body*, and we shall therefore speak more often in what follows of the virtual *centre* than of the *axis* for which, as far as the body is concerned, it is the representative.

It should be noticed that the only assumption which we have made is that the figure a has plane motion. The two points A and B were taken quite arbitrarily, with no conditions whatever as to the form of their paths. The result we have obtained is therefore perfectly general:—**Whatever be the motion of a figure in a plane at any instant it is always possible to find a point in the plane such that a rotation about it shall, for the instant, be the same as that motion.** At every instant the motion of the figure coincides absolutely with what it would be were the figure at that instant simply rotating about some particular point, a result which we may briefly express by saying that **the motion of every figure**

in the plane must be at every instant a rotation about some point in the plane. This point, the virtual centre, will always be denoted in our figures by the letter *O*.

Every point in the figure *a* must have the same motion (§ 3); if it were not so some points must move relatively to the others, which is impossible as long as the figure is rigid, which it is, by hypothesis. Every point in the figure, therefore, is turning about, that is, describing a circle about, the point *O*. The direction of motion of every point is therefore known, for it will simply be a tangent to such a circle, and therefore perpendicular to the line joining the point to *O*, or to what may be called the **virtual radius** of the point. The point *C*, for example (Fig. 16), is moving in the direction normal to *OC*, and *D* in the direction normal to *OD*.

Taking the converse of this we may say:—**When a figure is moving in any way whatever in a plane, the normals to the directions of motion of all its points, that is, the virtual radii of all its points, pass through one point, the virtual centre for its motion.** The virtual centre may therefore be defined, in geometrical language, as the locus of the intersection of all such normals, or, still more shortly, as the join of the virtual radii of all the points in the figure.

One special case requires to be mentioned here. The point-paths of *A* and *B* may be such that their tangents *a* and *b* are parallel (Fig. 17). The normals to their tangents,—the virtual radii—are therefore also parallel, and meet at no finite distance, however great. No virtual centre, therefore, can exist within any finite distance. It will, however, greatly facilitate the treatment of many parts of our subject if we treat parallel lines not as lines which never meet, but as

lines which meet at infinity. As we have already said (p. 14) we do not know what happens at infinity, but we find here, just as in the former case, that by making this assumption we are not led to any conclusions which are contrary to our finite experience. We shall therefore always treat parallel lines as lines which meet in a point

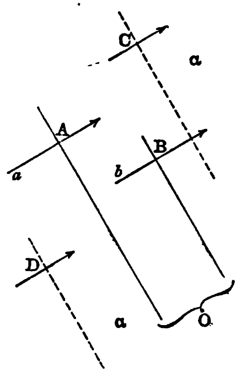

FIG. 17.

at infinity, that particular point at infinity which lies in their own direction. We shall find as we go on that such assumed points can be treated in every way similarly to points at a finite distance. The virtual centre of the body shown in Fig. 17 is therefore the point at infinity upon any one of the parallel lines normal to the direction of motion. The virtual radii of all points of the figure are parallel, so that at any given instant all points in the body are moving in the same direction, *i.e.* along parallel lines at right angles to their virtual radii, the point *A* along *a*, *B* along *b*, etc. In fact, the points are describing circles of infinite radius just in the way already described in § 2.

The figure α, then (Fig. 16), is virtually turning about the point O in the plane. That figure may be of any shape, and may, or may not, include the point O within itself. If it does so include O, then one of its own points must coincide with O, and the question arises, what motion has this point? As one point in a rigid figure it must necessarily have the same motion, as we have frequently seen, as all the other points in the figure, it must necessarily be turning about O. It is thus a point *turning about itself*, and therefore *not changing its position in the plane*. In one word, **the virtual centre of any figure relatively to a plane is always a stationary point of the figure.** Only one such point can be fixed, for to fix two would be to fix a line, and therefore to make the whole figure stationary. But one such point must always be fixed, and that point is always the virtual centre.

We have called the point O the virtual centre of the figure α relatively to the plane. But it is also the *virtual centre of the plane relatively to the figure*. For we have already seen (§ 3), that even when the motion of any two bodies α and β is of a quite general kind, the motion of β relatively to α is the same as that of α relatively to β. But here the motion of α relatively to β is a rotation about a particular and determinate point O. Therefore the motion of β relatively to a must be a rotation about the same point O. As we are dealing only with instantaneous motion, there is here no question of magnitude of the angle moved through. But it must be noticed (as at the end of § 3) that in order that the two bodies may occupy the same relative position at the end of the motion in each case, the *sense* of rotation about O must be reversed. That is, if the rotation of α relatively to β be right-handed, as in the figure, the rotation of β relatively to α must be left-handed.

Thus the virtual centre is always a *double point.* It is not only a coincident point in the two bodies whose relative motions it characterises, but is actually a point common to the two bodies, a point at which they may be supposed, for the instant, to be physically connected. If our ideal plane figures were replaced by the actual bodies which they represent, we might say that the virtual centre was a point through which might run the axis of a pin or shaft connecting the bodies, if only we had the physical means of instantaneously shifting the pin to suit each change in the position of the virtual centre.

We may therefore sum up the whole matter as follows, always assuming, as before, that we are only dealing with plane motions, and allowing the *centre* to represent the *axis,* for the reasons already given: **whatever be the real motions of any two bodies, they may be at any one instant fully represented by a simple rotation about a determinate point, which we call the virtual centre for the relative motions of the two bodies. This point is for the instant a point common to the two bodies—a point at which they may be supposed to be physically connected. This point is in each body one which has no motion (or is stationary) relatively to the other body.**

We shall find later on that the virtual centre is the point through which forces are transmitted from one body to another, and shall make much use of it in that capacity. The better to indicate its dual nature, we shall in connection with mechanisms most frequently use for it the letter O followed by a double suffix indicating the bodies for which it is the virtual centre; thus O_{ab} will stand for the virtual centre of a relatively to b, O_{ef} for that of e relatively to f, and so on.

In any combination of bodies, such as a mechanism, each body has some definite motion relatively to each of the others, and therefore has one point (in general a different point) in common with each. If in such a case we speak merely of "the virtual centre" of one of the bodies, without qualification, we shall always mean the virtual centre relatively to the particular body which is taken as fixed or stationary for the time being.

The student who finds any difficulty in following the reasonings of the last three sections, and especially of the present one, is very strongly recommended to make simple paper models for their illustration. The fact that the virtual centre is a fixed point in the plane often becomes clear when it is found that a needle can be stuck through the point which represents it without altering any of the conditions. A piece of drawing-paper should be cut out to represent the figure *a*, the paths *a* and *b* assumed, and the virtual centre found from them as in Fig. 16, and then the paths of *C*, *D*, etc., found by the help of *O*.

§ 8. PERMANENT CENTRE.

We shall find that in general a figure moving in a plane must be treated as having, at every instant, a rotation about a *different* point in the plane. But in one of the simplest and most important cases this is not so. The figure may have a simple motion of rotation continuously about one and the same point. Here we no longer have to substitute a rotation for an indefinitely short motion of some general kind and find a point about which the figure has *virtually* moved, but we have in the plane a point about which the figure is *actually* turning continuously, or at least for some

finite space of time. Such a point we should get, for instance, if we were considering the motion of a wheel on its shaft or a lever about its axis. To distinguish it from a virtual or instantaneous centre such a point is called a **permanent** centre. If we have only to do with a figure in one position at a time, and not with its motion through a series of positions, it is quite indifferent to us whether the point about which it is moving be a virtual or a permanent centre. Every proof or construction that avails for one can be used without the least modification for the other. We can in fact, by considering instantaneous motion only, reduce the most complex case, by a little trouble, to the level of the very easiest and simplest.

It may at first sight seem to be going a little out of the way to speak, say, of the "fulcrum" of a lever as its "permanent centre," and so to connect it closely with the somewhat more difficult notion of the virtual (and often, as we shall see immediately, non-permanent) centre. If the student will, however, take what is after all the very small amount of trouble necessary to master this latter idea, he will find himself amply repaid. Instead of having first to work at the theory of certain simple combinations, such as those known generally as the "mechanical powers," and then to find that all his work has to be done over again in some different and much more difficult method for all ordinary machines, he will find that he is able to work at the latter in just the same way, and using exactly the same constructions, as the former. The motion of the connecting rod of a steam-engine, for instance, to take a familiar case, is to all appearance a much more complex thing than the simple rotation of the crank. But by the use of the virtual centre all problems connected with it, kinetic as well as static and kinematic, become in every respect just as simple

and easy, and capable of treatment by just the same simple constructions.

With the certainty of gaining this end it is worth while to put the "permanent centre" here in its proper place as only a special case of the virtual or instantaneous centre, instead of separating it from the latter and treating it by itself without reference to the more complex, but quite as important, *virtual* rotations which are made by every part of every machine just as truly as the rotation of the shaft about its axis.

§ 9. CENTRODE AND AXODE.

In general, however, a figure moving in a plane (it must not be forgotten that such figures represent to us the actual *bodies* in motion in our machines, and are not mere geometrical abstractions) does *not* continue to move permanently about its virtual centre. On the contrary, if we take a number of different positions of a figure, as $A\ B$, A_1B_1, A_2B_2, etc. (Fig. 18), and find the virtual centre of its motion when occupying each position, we shall get, *in general*, as many different points for centres (O, O_1, O_2, &c.,) as we have taken different positions. In general, that is, the virtual centre is continually changing as the figure changes its position. If we suppose the figure to change its position continuously, that is, to occupy in succession a series of positions each one of which is indefinitely near to the one before it, the point O will also change its position continuously, that is, it will describe some continuous curve. This curve will contain *every* position of the virtual centre for the whole motion of the body; it will be, in geometrical language, the *locus* of the virtual centres. For

brevity's sake we shall call such a curve, when we have to speak of it, a **centroid**[1] or **centrode.**

We have not only supposed the figure *a* to be moving in the plane, but assumed also that it is its motion *relatively*

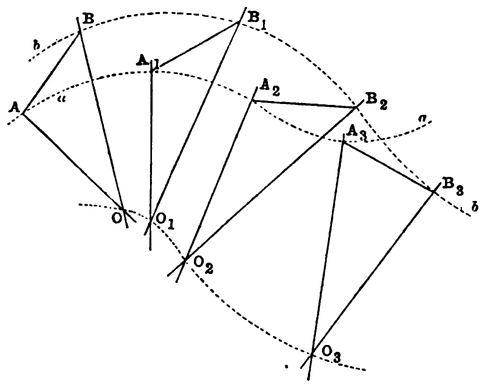

FIG. 18.

to the plane which has to be examined. The virtual centre is always a point *in the plane*, a fixed point, therefore, relatively to the figure. Keeping these points in view, the centrode may be more fully defined as **a curve in the plane which is the locus of the virtual centres of the motion of the given figure relatively to the plane.**

But we have seen that the virtual centre is always a double point, a point both of the body supposed fixed and

[1] This word was suggested to me by my late colleague, Professor Clifford, in connection with my translation of Reuleaux's *Theoretische Kinematik*. In the *Elements of Dynamic*, however, he afterwards used *centrode* for the same curve, and as this latter has since obtained wider use through its adoption by Prof. Minchin, I have decided to adopt it here.

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of that supposed moving. So now if we suppose the figure stationary and the plane moving about it—their relative motions remaining unchanged—we shall obtain for the locus of the virtual centres about which the plane is moving, a second curve, or centrode. This curve will be part of the fixed figure, just as the last was part of the fixed plane.[1] For any pair of bodies, therefore, having plane motion, we have a pair of centrodes, one belonging to (or forming a part of) each body, and each being the locus of the virtual centres, on the body to which it belongs, about which the other is turning. The pair of curves coincide always in one point the virtual centre for their motions at the instant.

As the relative motions of the bodies continue, the two centrodes roll upon one another, each one (along with the body of which it forms a part) turning relatively to the other at each instant about the point which at that instant they have in common. But a body cannot turn about two different points at one time, so that the rolling of the centrodes, by fixing this point at each instant, uniquely determines the motion of the bodies. If therefore we are given a pair of centrodes for two bodies, we have all the data necessary for the complete determination of their motions, that is, for finding all the possible relative positions which they can occupy, assuming, that is, the geometrical or physical possibility of rolling the one curve accurately on the other without slipping. Figure 19 illustrates this: AA_1 and BB_1 are two bodies having plane motion, for which a and b are respectively the centrodes, in contact at the present virtual centre. By taking (say) a as fixed, and rolling b upon it, we find any number of positions of the line BB_1

[1] There is, of course, no real difference here between "figure" and "plane"; each equally represents to us a solid body. The difference of phrase is only retained for convenience of expression.

(some of which are dotted), and therefore (§ 4) of the plane figure or body which that line represents.

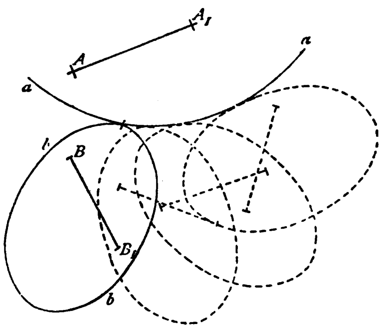


FIG. 19.

We said that the centrodes *rolled* upon one another. As they have always one point in common, they must either roll or slide upon one another. But sliding at their coincident point is inconsistent with the essential characteristic of that point, viz., that it is for the instant fixed, all others turning round it. We were therefore justified in saying that the motion was rolling, and can say further that the common point of the two curves is a point of contact, a point where they have a common tangent.

We have seen that the virtual centre of a figure may be regarded as simply the trace, upon the plane, of the virtual axis about which the body represented by the figure rotates. Looking at the centrode from this point of view, every point in it represents a line, and the curve, or locus of points, determined by the motion of the figure in the plane, becomes—for the motion of the body of which that figure is a section—a surface, or locus of lines, each one an instantaneous axis for the motion of the figure. All these lines

are parallel, as they are all at right angles to the plane of the centrode. Any surface consisting in this way of straight lines is called a **ruled surface,** and where all the lines are parallel, a **cylinder,** taking the latter in its general meaning, and not in the common restriction to a parallel ruled surface of *circular* section. As this particular surface is made up of lines which have a special importance to us as axes of rotation, it is called an **axode**—the locus of the virtual axes for the motion of the body with a plane section of which we were formerly dealing. For the same reasons, however, which make it convenient rather to speak of the *figure* than of the *body* which it represents, and of the virtual *centre* rather than of the *axis* of which it is a projection, we shall, in considering plane motions, use rather the *centrode*, or locus of centres, than the *axode*, or locus of axes, when we have occasion to deal with either. Here as before, however, it is of importance for the student to bear in mind the way in which the two are connected, and that we choose the more limited and simple notion to represent the more general and complex one, solely as a matter of convenience. If we were dealing with spheric instead of plane motions we should find it probably more convenient to use the axis and axode than the spheric curves which there take the place of the centrode, and in problems connected with twist, and other still more general forms of motion, we can no longer obtain either centrode or axode, the place of the latter being taken by a complex ruled surface, which has by no means the simple form or meaning of the axode for plane or spheric motion. These complex problems of general motion, fortunately, rarely occur in practical work; and in this book we shall not enter into their consideration further than to point out, at the proper time, the general direction of the methods which may be employed in their solution (see Chapter XI.)

CHAPTER III.

THE CONSTRAINMENT OF PLANE MOTION.

! 10. ELEMENTS OF MECHANISM.

HAVING so far investigated certain questions relating to the nature of plane motion, we have now to examine the nature of the means used to obtain such motion, *constrained*, in machines. We have already pointed out (p. 8) that the motion of any piece of a machine is determined by the *form* of its connections with the other pieces, assuming these connections to be of suitable material. This form has to be such as not only to allow the required motion, but absolutely to render impossible all other motions in the way which has been already explained in § 1. The principle of the method used to obtain this double object is as follows, supposing it applied to a perfectly general case. We first form some part of one of the two bodies whose relative motion is to be constrained, into any convenient shape, say such a projection as A on the body a, Fig. 20. Then, bringing the other body β to rest, we find all the positions of the shaped portion of the first relatively to it, and the curves bounding these positions form a figure B traced out on β by A, which is called in geometry the *envelope* of A upon β. By now removing the material of β within this figure so

as (in this case) to make a curved slot or groove in β, bounded by the lines shown in the figure, we could allow the projection A to lie in the slot B, and should in this way have made a connection between the two figures which would fulfil the first condition, namely, the permitting of the required motion to take place. It would not, however, necessarily fulfil the second condition, namely, the prevention of all other motions. It is evident, in the first place, that the

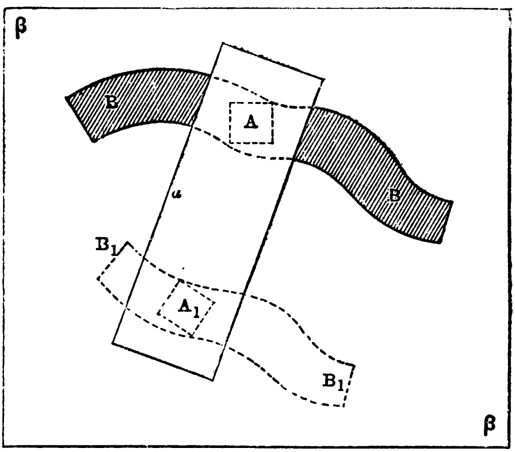


FIG. 20.

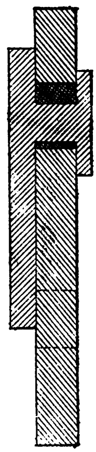


FIG. 21.

two bodies could be separated at will by being pulled right apart at right angles to the plane of motion. This disturbance is prevented by giving to A and B such a **profile** (or section perpendicular to the plane of motion) as may render this motion impossible; as, for instance, by carrying A right through B (as shown in section Fig. 21), and attaching a collar to its inner end. This of itself, however, does not *necessarily* constrain the motion completely, for it is quite

possible that in some places the corners of A may be quite clear of B, and the motion therefore left quite uncontrolled. In order to rectify this, if it occur, either another form must be adopted for A, and therefore for B, or else some other piece, A_1, must be placed on α, with its corresponding envelope B_1 on β, in such a way that the one completely constrains the motion in every position where it is left partially free by the other. This can be done by the application of certain rules which we need not examine here.

A pair of such forms as those we have supposed to be placed on the bodies α and β, when they are arranged so as to make the motion completely constrained, are called a **pair of elements**, or more fully, a **pair of kinematic elements.** It is seen at once from their nature that these elements occur necessarily in pairs, and never singly. A single element can no more constrain motion than a single body can make a machine (§ 1); they must always go in pairs, and these pairs of elements form the lowest factor to which we can reduce a machine.

We have supposed for our illustration a very general case indeed, and one that occurs very seldom, although it does sometimes occur, in practical work. Most of the pairs of elements which we find in machines are of a very much simpler kind than that shown in Fig. 20.

Of these simpler forms the two most important are those continually employed in machines to constrain the two special forms of plane motion which we examined in § 2, rotation and rectilinear translation. These may be called, on account of the motions which they constrain, the **turning** and the **sliding pairs** respectively. The former takes the shape of some solid of revolution, having such a profile as to prevent axial motion ; in its commonest form it is the cylindric pin and eye of Fig. 22, where the collars upon

the pin prevent the axial motion. The sliding pair is in form essentially *prismatic*, that is, it is a solid having plane sides, parallel to the direction of motion. It commonly takes in machines some such form as that shown in

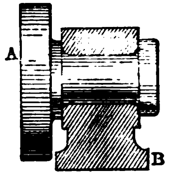


FIG. 22.

Fig. 23—a bar and a guide, or a slot and a block. The profile of the elements in each case is such as to prevent any motion *across* the required direction, just as in the turning pair.

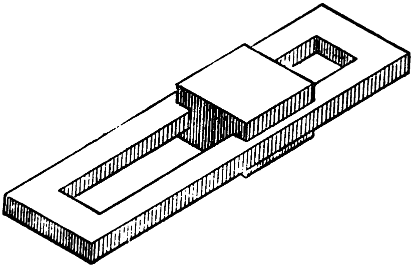


FIG. 23.

Two characteristics of these particular pairs render them specially valuable to engineers. *Firstly*, they are very easily made. The production of circular surfaces is probably the most easy operation with which the engineer has to do, and the lathe, the machine in which this operation is carried on,

is the most common of all machine tools. Next to the production of turned or bored surfaces, that of flat surfaces is the operation most readily performed—the planing or shaping machines used for the purpose are always at hand. *Secondly*, the contact between the two elements in each pair is a *surface* contact. In the general case (shown in Fig. 20) the element AA_1 only touched BB_1 along, at most, three or four lines, but in the turning and the sliding pairs contact exists over a considerable surface in each element. From a geometrical point of view the constrainment is equally good in both cases; but, looked at as part of a machine, we have to keep in mind that the surfaces will wear, and we must consider that constraint the most perfect which is likely to be least disturbed by the wearing away of the constraining forms. From this point of view that form of element is best in which the pressure is distributed over the largest area, and contact over a surface is always more advantageous than contact only along a line or a few lines. Pairs of elements working with *surface* contact are called **lower pairs**; all others having *line* contact may be distinguished as **higher pairs.**

There are only three classes of surface with which this surface contact, during motion, is possible. These are (1) plane surfaces, (2) surfaces of revolution, and (3) cylindric screw surfaces. The first is utilised in the sliding and the second in the turning pair, the third (Fig. 24) is utilised in a **twisting pair** of elements (of which the common screw and nut form the most familiar example) of which we shall have to say something further on; the motion constrained by the latter is not *plane* and therefore does not fall to be considered here. The only plane motions, therefore, which can be *constrained directly* by elements having surface contact, are rotation and rectilinear translation. For all other plane

motions we must have recourse either to higher pairs of elements, with line contact only, or to indirect constraint

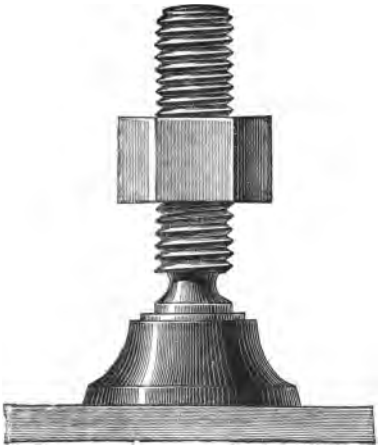


FIG. 24.

with lower pairs in the way indicated in the following section.

§ 11. LINKS, CHAINS, AND MECHANISMS.

We have seen how, in order to constrain the motion of one body relatively to another, it is necessary to connect them by a suitably-formed pair of elements. Two bodies thus connected form the simplest combination which we can treat as a machine,—but by our definition two such bodies may actually, as they often do, form a machine. We have now to look at the way in which more complex machine forms are built up from this very simple beginning. In the

case supposed each body carried one element only; and with this limitation nothing more can be obtained. To go further, that is, to combine more than two bodies into a machine, each one must have at least two elements forming part of it, and the number can be increased indefinitely. For the present let us see what can be done with bodies each containing not more than two single elements.

One essential condition of the motions in any machine, and therefore in the combination of bodies, which we now wish to make, is that at no instant shall it be possible for any one of the bodies which form it to move in more than one single way. If any alternative motion were possible at any instant, the particular motion occurring would be determined by the direction and magnitude of the particular forces causing motion at that instant. This condition is impossible—that is, it must be made impossible in a machine, in order that its fundamental conditions as to constraint may be complied with. The same condition may, for convenience sake, be stated somewhat differently, namely, thus—**It is essential that among all the bodies which form a machine, and in which motion is possible at a given instant,[1] no one should be able to move without all the others undergoing certain definite changes of position also.** For if at any instant the bodies *a*, *b*, *c*, etc. in some machine are movable, and if *a* and *b* can either move or not move while *c* is moving, it is only a question of the nature of the moving force whether *c* move alone or whether it carry *a* and *b* with it. But the relative position of *c* to *a* and *b* is of course quite different if the latter move, to what it would be were

[1] This limitation is necessary because in many machines there are certain bodies which can only move periodically, being held fast by special contrivances when they are not required to move. (See § 60.)

they to remain stationary, and, under the circumstances supposed, the position of *c* at a particular instant relatively to *a* and *b* would depend entirely upon the forces acting on *c*, and could be altered altogether by a change in those forces. By definition, therefore, the motion of *c* relatively to *a* and *b* would have ceased to be constrained, and a combination such as has been supposed could not form part of a machine.

The motion of every body which forms part of a machine must be constrained relatively to all the other bodies constituting the machine. This is a proposition so obvious that we may simply state it without proof.

Bearing in mind these conditions, we can now go on to examine the way in which a machine can be built up of bodies each containing not more than two elements. The question comes at once, Can a machine contain bodies of single as well as of double elements? It cannot; for a body having only one element can only have its motion constrained to the one body to which it is paired. Such a body cannot therefore form a part of a machine containing more than one other body, for its motion relatively to any other bodies would be quite unconstrained. Our present work, then, may be limited to an examination of the ways in which bodies containing two elements each can be combined into machines.

Bodies, such as we have now to consider, which are arranged to form part of a machine by being provided with two or more suitably-formed elements, are called, when they are looked at merely in reference to the motions of the machine, **kinematic links**, or simply **links**. In order that a series of links may be combined into a machine it is necessary, of course, that the elements which they carry should be such as, when connected in pairs, give the required motions. In

order, further, that the proper pairing may take place, the two elements on any one link must not belong to the same pair, but the links must be so arranged that (calling them *a*, *b*, *c*, &c.) one element on *a* shall pair with one on *b*, the second on *b* with one on *c*, the other on *c* with one on *d*, and

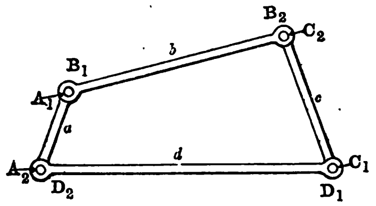


FIG. 25.

so on. On the last link there will then be one element left unpaired, while only one of those on the first (*a*) has been paired. These two elements must then be paired together, and the arrangement is complete. Figs. 25 and 26 show this for two very simple cases where the pairs used are all either

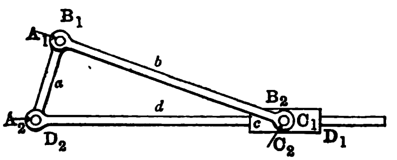


FIG. 26.

turning or sliding pairs. Four links *a*, *b*, *c*, and *d*, are used in each case, carrying respectively the elements[1] A_1A_2,

[1] Here and elsewhere the collars or flanges necessary for preventing cross motion in the pairs (see p. 54) are omitted in the figures wherever their insertion might tend to impair the clearness of the illustrations.

B_1B_2, C_1C_2, and D_1D_2; a is connected with b by the pair, A_1B_1, b with c by the pair B_2C_2, c with d by C_1D_1, and then there are left D_2 and A_2 to form a fourth pair. These being connected, the combination is finished.

A series of links completely connected in this way—connected, that is, so that no element is left single, but each one paired with its partner,—is called a **closed kinematic chain**, or simply a **chain.** Each link in the chains of Figs. 25 and 26 is paired directly with two others, its **adjacent** links. Its motion relatively to each of these is therefore completely constrained by the pair of elements

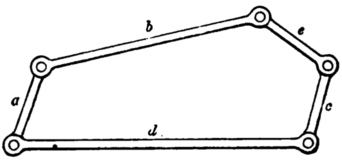


FIG. 27.

which connect them. But its motion relatively to non-adjacent links must equally be constrained, and that this is the case cannot be assumed without proof. For although the constrainment is really complete in the particular chains shown, it would have been just as easy to construct a chain in which it would not have been complete. Figure 27 shows such a chain. It does not differ much in appearance from that of Fig. 25, and has been put together by exactly the same method, but yet it is a totally useless combination of links, while the other is among the most familiar chains in existence. To prove that the (plane) motion of a body is constrained we know that we need only to prove the constrainment of two of its points (p. 34).

The motion of the whole body is that of any line in it, so that if one line have its motion constrained the whole body must also have constrained motion. If, on the other hand, it can be shown that even one point of the body is unconstrained the motion of the whole body must also be unconstrained. It is not always easy to prove that no point in a particular body, or that not more than one of its points, is constrained. But there is seldom much difficulty in showing either that *two* points *are* constrained, or that *one* point is *not*, or else that two of the moveable links might be made into one, in any of their relative positions, without destroying the movability of the rest of the mechanism. Any of these three conditions would be sufficient to settle the matter

In the cases before us let us take first the chain of Fig. 25. Can it be proved, for example, that the motion of the link b is constrained relatively to the non-adjacent link d? We know that the motion of every point in the links a and c is constrained relatively to d; but b has one point *in common* with each of these links, viz. its virtual centre relatively to each of them (p. 45). If these points be *permanent centres*, *i.e.* if they retain always the same position on a and c respectively, then their motion would be constrained, and hence the motion also of b, as they both belong to that body, would be also constrained. In Fig. 25 this is the case—b moves relatively to a about the centre of the pair A_1B_1, and relatively to c about the centre of the pair B_2C_2. Both centres are permanent, and the motion of b relatively to d is therefore constrained. The motion of d relatively to b is therefore constrained also. By precisely the same reasoning it can be proved that the relative motions of a and c are also constrained. Here, however, in Fig. 26, there is the difference that the virtual centre of c

relatively to d is not a point on the pair—the centre of a turning pair—as in the last case. It is the point at infinity (p. 43) in the direction perpendicular to the motion of c on d. It is, however, *the same point* for all positions of c on d, and must therefore be treated as a permanent centre just as fully as the visible centre of the pair B_2C_2. Figs. 25 and 26 therefore show *completely constrained* kinematic chains.

In more complex chains, such as that shown later on in Fig. 28, the proof of constrainment is not so simple, but can be handled in exactly the same way. In such a case the points corresponding to the centres A_1B_1 and B_2C_2, are not always themselves moving about permanent centres, but about points whose constrainment has first to be proved. We shall find that this is always very easy to do.

The case is quite different with the chain shown in Fig. 27, in which a fifth link, e, is added between c and b. Examining the motion of b relatively to d, we see at once by reasoning similar to that given above, that *one* point is constrained, namely the centre point of the pair connecting a and b. The virtual centre of b relatively to c is, however, no longer a permanent centre, but a moving point whose constrainment has to be proved. This will be found impossible, however, without the assumption that either a or c is fixed as well as d. It will be found further that either a or c *could* be fixed as well as d, while still the remaining three links would remain movable, the chain, in fact, becoming identical with that of Fig. 25. This contradicts the fundamental condition (p. 59), that it shall not be possible for any link to be moved without all the other movable links moving also. Having thus proved that the relative motion of one link relatively to any other is unconstrained, it is unnecessary to examine the motions

of other pairs of links—we may say at once that the chain is not a constrained one, and cannot therefore, in its present form, become part of a machine.

We have now before us a kinematic chain completely constrained; in other words, a combination of bodies so connected that every motion of each relatively to every one of the others is absolutely determinate, independent of external force. The step from the chain to the machine is a very simple one. The chain in itself only constrains the motions of its links relatively to each other; the motions of the different parts of a machine must be constrained relatively to some definite standard, as, for instance, the earth (see § 3). To convert the chain into the machine, one of its links must therefore be fixed relatively to the earth or other standard. The motions of the remaining links are then constrained relatively to the same standard, and the problem is solved.

Any chain having one link fixed might be called a machine, and essentially it is one. But it is convenient to have some word to distinguish the ideal machine, such as is shown in our engravings, with its straight bars and regularly shaped blocks, from the actual machine of the engineer with its complex masses of iron and steel. In their motions the ideal and the actual machines are identical, in all dynamic problems also the one can represent the other, but still there is so great an apparent difference between them, that in common usage the former is called generally a **mechanism**, and the word **machine** is reserved exclusively for the latter. Using, then, this established nomenclature, we can put down the conclusions at which we have arrived in the form of the following propositions:—

We obtain the simplest combination having the nature of a machine by connecting two bodies of

suitable material by such geometric forms as completely constrain their relative motions :—

These constraining forms are called elements, and can only occur in pairs. If contact between the two elements of a pair exist only along a line or a limited number of lines, it is called a **higher** pair, while pairs which have *surface* contact are called **lower** pairs. Two kinds of lower pairs only are available for the constrainment of plane motions, these pairs being called **turning** and **sliding** pairs respectively, from the nature of the motion which is constrained by them :—

Where a constrained combination is made of more than two bodies, each one must carry at least two elements, belonging to different (although possibly congruent)[1] pairs. Such bodies are called links. Lastly,—

A series of links so connected that each element in each is paired with its partner in another, and further so that the motion of every link is constrained relatively to that of every other, is a kinematic chain, and by fixing one of the links of such a chain relatively to the earth (or other standard) it becomes, finally, a mechanism. A mechanism is the ideal form of a machine, and represents it fully and absolutely for all our problems.

The form and position of the elements of any link determine its motions; the shape of the body of the link itself is quite immaterial, so long as it is not such as to foul any of the other links during its motion. In practice links of similar machines take the most widely different forms in different cases, and very frequently this form bears no resemblance to the

[1] See p. 17.

skeleton form of the corresponding link in the mechanism, the elements in the corresponding links being, however, identical. The mechanism of Fig. 26 is that which appears, for instance, in an ordinary horizontal steam engine. The link *a* becomes the crank of the engine, which in form it resembles: the link *b* becomes the connecting-rod, not quite so similar in form: the link *c* of the mechanism becomes in the machine the crosshead, piston-rod, and piston: and the link *d* the cylinder, the frame or bedplate with its crosshead guides, and the plummer block for the main bearing. The two last links have, therefore, in their ideal form, scarcely any resemblance to their counterparts in the actual machine. In another case, as we shall see, the link *b* is used as a cylinder and *d* as a piston; in another *a* becomes a fixed cast-iron framing, and so on; it is unnecessary to multiply examples. In every case the motions of the bodies forming the machine are determined by the nature of the elements connecting them, and are unaffected (under the limitation stated above) by the form of the bodies themselves.

Of course this holds good equally for the case where only two bodies are used, connected by a single pair of elements. The form of the bodies may be anything whatever, provided it is not such as to hinder the motion, so long as the elements themselves are correctly designed; the motion is determined by the latter only, and is quite unaffected by the former.

We have seen that a mechanism is formed from a chain by fixing one link of it. But *any* link of the chain may be fixed; no one link is in this respect different from the others. Hence **we can obtain from any chain as many mechanisms as it has links.** In such a case as Fig. 26 the four mechanisms which could be thus obtained would all be different; but this is not always the case, two or more

of them may be similar or identical. Their total number, however, must always be equal to that of the links in the chain. The alteration by which we change one mechanism into another, using the same chain, the change, that is, in the choice of the fixed link, is called the **inversion** of the chain.

We have spoken in this section exclusively of chains whose links each contain only two elements. Such chains are called **simple chains**, and include very many of the most important mechanical combinations existing. But what

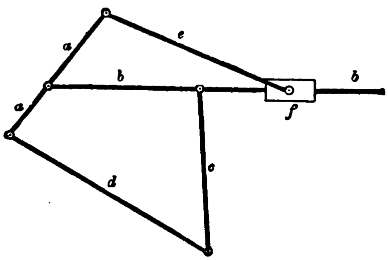


FIG. 28.

has been said about them applies, with only verbal alterations, to **compound chains**, or those chains which have some links containing more than two elements. Such a chain is shown in Fig. 28, which is a chain containing six links, two of which, *a* and *b*, have each three elements. Compound chains, and the mechanisms formed from them, do not differ essentially from simple chains and mechanisms. Naturally they are a little more difficult to deal with,—nothing more. We shall have occasion to examine several of the more important of them further on.

We must now proceed to apply to the mechanisms whose nature we have been examining the principles of "virtual" motion, which we sketched out in former sections.

CHAPTER IV.

VIRTUAL MOTION IN MECHANISMS.

§ 12. DETERMINATION OF THE VIRTUAL CENTRE IN MECHANISMS.

WE have seen that in order to determine the virtual centre about which a body is moving at any instant, it is necessary and sufficient to know the direction of the motion of two points in the body at that instant. We must now consider this more in detail, in order particularly to apply our knowledge to the solution of the problem in the case of mechanisms.

The path in which a point is moving in the plane may be supposed given, either by its equation or by its form actually traced out on paper. In the former case the direction of motion, or tangent to the curve, can be calculated, and in the latter case it can be drawn. We have to deal exclusively with the latter case in our work. There are few cases in which it is at all difficult to draw the actual path in which any point of a mechanism is moving, and to construct a tangent to this path at any point, and no cases at all, so far as we know, in which it is not greatly more convenient to do so than to calculate an equation to that path. Finding the

direction of motion of a figure then means, for us, simply drawing the paths of two of its points and constructing tangents to them, or of course (if possible) constructing the tangents without drawing the point-paths themselves, which are not, in most cases, of any direct importance to us.

We require to know the direction of motion of two points in the body. *Any* two points will serve, provided their virtual radii be not coincident, in which case, of course, they would not determine the intersection which we require. The problem, therefore, resolves itself simply into a choice of points, a matter which we must here examine briefly

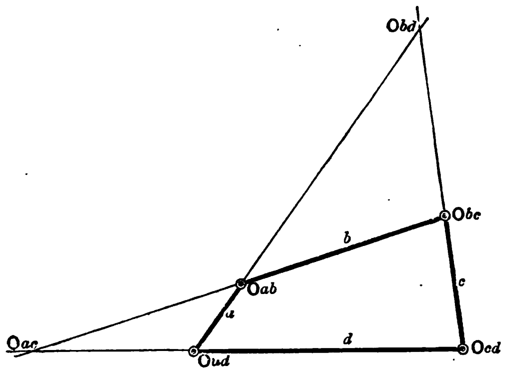


FIG. 29.

—we shall have numerous examples in succeeding chapters. To start with the simplest possible case, let it be required to find the virtual centre of every link relatively to every other in such a mechanism as Fig. 29, which consists of four links connected by four turning pairs. The axes of the pairs are all parallel, so that the links have only plane motion. Calling the links *a*, *b*, *c*, and *d*, we shall call

their virtual centres O_{ab}, O_{bc}, etc., the suffixes denoting the links for which the particular point O is the virtual centre. The virtual centres of *adjacent* links are permanent centres, and are, as we have already seen, simply the centres of the pairs connecting them. Relatively to d for instance, every point in a moves always about the point O_{ad}, which is the centre point of the pair connecting a and d. By mere inspection therefore, we have at once the points O_{ab}, O_{bc}, O_{cd} and O_{da}, as the virtual (and permanent) centres of the four pairs of adjacent links. There are two other virtual centres in the mechanism, those for the two pairs of non-adjacent links; O_{ac} for the links a and c, and O_{bd} for the links b and d. We may take the latter first;—b is connected to d, and its motion constrained, by the links a and c, and we know the motion of every point in these two links relatively to d. But b has one point in common with a, viz. the point O_a ; O_{ab} and also one point in common with c, the point O_{bc}. We know the motion of these points relatively to d as points of a and c, and of course they must have the same motion relatively to d as points of b, for they cannot have two different motions relatively to the same body at the same time. We have therefore at once the motion of two points in b relatively to d, which is all we require. O_{ab} is moving in a circle round O_{da}—without drawing its path then, or even constructing the tangent to it, we can at once draw its virtual radius, which is at right angles to the tangent, and which is simply the axis of the link a. In exactly the same way the axis of c is the virtual radius of O_{bc} and is at right angles to the direction in which it is moving. The virtual centre of b relatively to d is therefore at the join of these two axes, as shown by the fine lines in the figure. By the same reasoning it can be shown at once that the point O_{ac} is at the join of the axes of b and d.

Quite generally, therefore, in a chain such as Fig. 29, consisting of four links connected by four parallel turning pairs, **the virtual centre of either pair of opposite links is the join of the axes of the other pair; the virtual centre of any pair of adjacent links is the join of their own axes, and is a permanent centre.**

An inspection of Fig. 29 shows a rather remarkable regularity in the disposition of the virtual centres. The six centres lie in threes upon four lines, and the three centres on

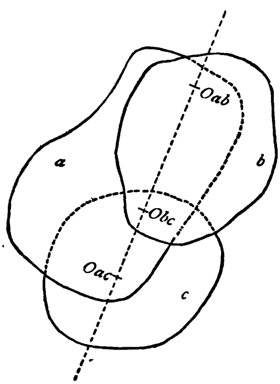


FIG. 30.

any one line are always those corresponding to three particular links out of the four. The links *a*, *b*, and *c*, for instance, give us the three virtual centres O_{ab}, O_{bc}, and O_{ac}, and these three points are in one line, here the axis of *b*. The links *b*, *c*, and *d*, similarly, give us the points O_{bc}, O_{cd}, and O_{bd}, and these again lie all on one line, here the axis of *c*. The question comes at once whether this is some mere coincidence, belonging to the very simple mechanism which we have chosen for illustration, or whether it represents some general law which we may apply in other cases. It is in

fact quite general, and the proof is simple. Let *a*, *b*, and *c* (Fig. 30), be any three bodies whatever, having plane motion, and let O_{ab}, O_{bc}, and O_{ca} be the virtual centres for their motion. O_{ac} is a point both of *a* and of *c*; as the former it is moving about O_{ab} relatively to *b*, as the latter about O_{bc}. That is, its direction of motion as a point in *a* is at right angles to the line joining it to O_{ab}, and its direction of motion as a point in *c* is at right angles to the line joining it to O_{bc}. But it can have only one direction of motion relatively to *b*, whether it be treated as a point of *a* or of *c*, and as this direction is normal to both the lines just mentioned, they must either be parallel or coincident. They cannot be parallel, for they both pass though the same point O_{ac} on the paper—they must therefore coincide. The radius O_{ac} O_{ab} coincides with O_{ac} O_{bc}—the three points named therefore lie in one straight line. We might have started with O_{ab} or O_{bc} instead of O_{ac} and should always have come to precisely the same result, which may be summed up as follows; **If any three bodies *a*, *b*, and *c*, have plane motion, their three virtual centres O_{ab}, O_{bc}, and O_{ac} are three points upon one straight line.**[1]

We may now examine another simple chain, the one shown in Fig. 31, which is the same as one which we have already noticed. Using the same notation as before, we have again the points O_{ab}, O_{bc}, O_{cd}, and O_{ad}, the virtual centres of adjacent links, at once. Three of them are, as in the last case, the centres of turning pairs, and the fourth is the centre of the sliding pair *c d*, and therefore a point at infinity. All four, including, O_{cd} (see pp. 43 and 46),

[1] This proof, it may be noticed, is quite independent of the bodies being adjacent links in a mechanism, or indeed of their belonging to a mechanism at all; it applies to any constrained *plane* motion. For the corresponding theorem in spheric motion see § 63.

are permanent as well as instantaneous centres. The virtual centres of opposite links, O_{bd} and O_{ac}, are as easily found as in the last case, but the points which determine them are not, perhaps, quite so obvious. The link b has, as before, one point in common with each of the links a and c; we know the motion of every point in these links relatively to d, for we have found O_{ad} and O_{cd}, we therefore know the motion of two points in b relatively to d, these points being

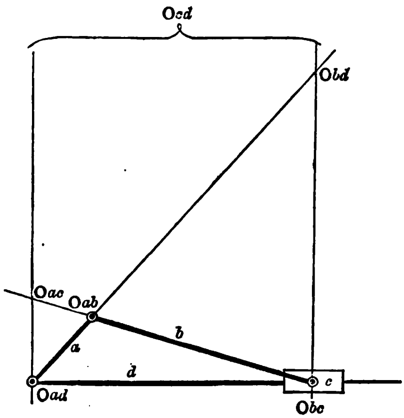


FIG. 31.

O_{ab} and O_{bc}. The virtual centre of b relatively to d is at the join of the virtual radii of these points, exactly as in the former case. The virtual radius of the point O_{ab} is the line joining that point to O_{ad}, or the axis of the link a. The virtual radius of O_{bc} is the line joining that point to O_{cd}, which is simply a line perpendicular to the axis of the sliding pair. The construction lines and the point O_{bd} at their join are shown in the figure.

By similar reasoning we get the point O_{ac}, the virtual centre of *a* relatively to *c*, but the reasoning is perhaps a little more difficult to follow. The link *c* has two points whose motion relatively to *a* we know, for it has one point in common with each of its adjacent links *b* and *d*, both of which are adjacent to *a*. O_{ac} must be, as before, the join of the virtual radii of these two points. The virtual radius of the one, O_{bc}, is the line joining it to O_{ab}, or simply the axis of the link *b*, and can at once be drawn. The virtual radius of the other, O_{cd}, is the line joining O_{ad} and O_{cd}. But O_{cd} is a point at an infinite distance, hence all lines passing through it are parallel on our paper (p. 43), so that to draw the line in question we have only to draw through O_{ad} a line parallel to the virtual radius of O_{bc} already constructed, (and therefore perpendicular to the axis of the sliding pair) and we have the line required, which gives us O_{ac} directly. The construction is shown on the figure.

By the help of the theorem about the virtual centres of three bodies which we proved above, this proof can be much shortened. From the fact that *a*, *b*, and *c* are three bodies having plane motion, we know that the point O_{ac} must lie on the line joining O_{ab} and O_{bc}, and similarly, considering the three bodies *a*, *c*, and *d*, we know that O_{ac} must lie on the line joining O_{ad} and O_{cd}. To find O_{ac} therefore, we have only to draw these lines to their join, which is just what we have done. Similar reasoning would, equally briefly, have given us the position of the point O_{bd}.

As these mechanisms are very important and will often be referred to, it may be well to use the name "lever-crank" for Fig. 29, and "slider-crank" for Fig. 31, the link *d* being supposed the fixed one in each case.

Fig. 32 shows a mechanism having a very close relationship to Fig. 31, but one in which it may appear at first sight

to be a very much more difficult matter to determine the virtual centres. The difficulties which occur in this case are more apparent than real, but as they occur more or less frequently, it will be quite worth while working through the determinations in detail, and then trying to find out what the real differences are between the three mechanisms shown in Figs. 29, 31, and 32.

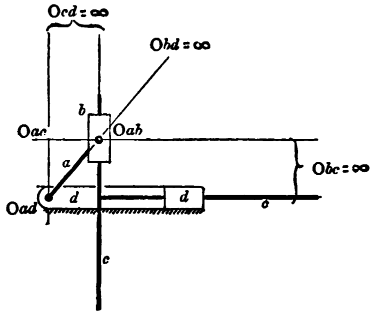


FIG. 32.

In Figs. 31 and 32 the links *a* and *d* are exactly the same, and also the pairing of *d* with *c*, and of *a* with *b*. But the link *c* now (Fig. 32) consists of two sliding elements instead of one sliding and one turning, while the link *b*, being paired to *c*, takes a form similar to that which *c* had before, viz., one turning and one sliding element. The virtual centres O_{ab}, O_{ad}, and O_{cd} remain exactly as before,—we may proceed to find the others. Take the point O_{bc} first: the link *b* simply slides upon *c*, their virtual centre must therefore be a point at infinity, and the direction of that point must be at right angles to the known direction of the sliding or translation of *b*. The position of O_{bc} must therefore be as shown in the figure. The point O_{ac} is equally easily found: it must lie upon the line containing O_{ad} and O_{cd}, and also

upon the line containing O_{bc} and O_{ab}, and therefore must be at the join of these two lines at the point marked. We come lastly to the virtual centre O_{bd}. This must, in the first place, be on the line joining O_{ad} and O_{ab}, the axis of the crank. It must also lie upon the line which joins O_{cd} and O_{bc}. But *both* these points are at an infinite distance, where then *is* the line joining them? That we know no more than we know where the points themselves are (see p. 14). But reasoning from known properties of lines within our reach, we conclude that such a line must be *entirely* at infinity, *i.e.* that it must be treated as having *all* its points at an infinite distance. So therefore we say that the point O_{bd}, being a point upon a line which has already two points at infinity, must be itself necessarily at infinity also, and as its direction is already known, the point itself is completely fixed, as marked in Fig. 32. It follows, of course, that *b* should have only a motion of pure translation (p. 15) relatively to *d*, and this it can readily be proved to have.

A closer examination of Figs. 29, 31, and 32 will help to show some interesting relations between the mechanisms represented in them. In Fig. 29 we saw that all the six virtual centres lay in threes upon four straight lines, the axes of the four links. In Fig. 31 the six virtual centres also lie in threes upon four lines, but at first sight these do not appear to be the axes of the links, for the apparent axis of *d* (the lower horizontal line in the figure) is not one of them, while two of them (the two vertical lines) are not apparently axes of links at all. The discrepancy is only apparent. If we could construct a mechanism which, having *a* and *b* exactly as in Fig. 31, had *c* and *d* stretching away from their end-points to a point O_{cd} *at infinity*, and there connected by a pin, we should have a mechanism (see § 52) which might be called an ideal form of that shown in Fig. 31. Its links

would have exactly the same relative motions—the point O_{bc} and all other points of the new link *c*, would move in straight lines parallel to the line *d* in the figure, because they would be turning about a point at an infinitely great distance, while its four links would be connected simply by four turning-pairs as in Fig. 29. And the four lines which would then form the axes of the mechanism are exactly those upon which the virtual centres do lie in threes. Those lines, therefore, may be considered to be the ideal axes of the links in the mechanism. But as we cannot construct a pin-joint at infinity we have to content ourselves with imitating the motions to be obtained from it by the use of a sliding pair. Kinematically, therefore, the mechanism of Fig. 31, the slider-crank, may be said to be the same as that of Fig. 29, with the links *c* and *d* made infinitely long.

By precisely similar reasoning, which it is unnecessary here to repeat at length, but which the student will find it a very useful exercise to write out in full, it can be shown that the mechanism of Fig. 32 is derived from the slider-crank by making the link *b* infinitely long. Of the four ideal axes only one (that of *a*) coincides with the actual axis of a link, two of the others are lines parallel to the arms of the link *c* and passing through the points O_{ad} and O_{ab} respectively, while the last is the line at infinity to which we have already referred. Fig. 32, therefore, represents the constructive form taken by a slider-crank with an infinitely long connecting-rod, a mechanism frequently referred to for practical purposes. In § 52 these points will be found more completely treated from a somewhat different point of view.

The use of the theorem that the virtual centres of three bodies having plane motion are three points upon a line, greatly shortens the proof in some cases of simple mechanisms, but such cases can generally, if not always, be proved

without it. It is, however, indispensable when we have to find the virtual centres of links in compound chains. One example of this case will suffice for the purposes of the present section. We may take the compound chain of Fig. 33, which is similar to the one shown in Fig. 28. Here

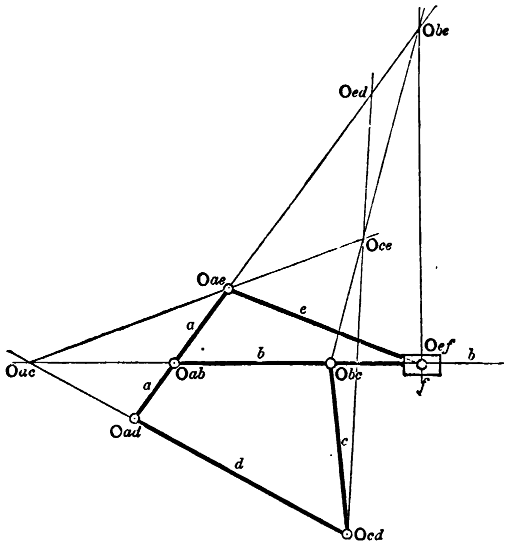


FIG. 33.

there are in all fifteen virtual centres lying in threes upon twenty lines, some of the latter, however, being coincident. The virtual centres of adjacent links can be marked at once, just as in a simple chain, and of the rest we may pick out three simply to illustrate the method of dealing with such cases. Let it be required to find the virtual centres of *c* relatively to *e*, of *a* relatively to *c*, and of *d* relatively to *e*. We see by inspection that we have already a

line which contains O_{ab} and O_{bc} and which must therefore contain O_{ac}, one of our required points. But we have also a line containing O_{ad} and O_{dc}, upon which, again, O_{ac} must lie. The point O_{ac} must, as it is upon both these lines, be their join, which therefore can be marked at once. So far we have nothing different from a problem already solved (p. 71), the links *a*, *b*, *c*, and *d* forming by themselves a mechanism the same as that of Fig. 29. The position of the point O_{ac} is not affected in any way by the additional links *e* and *f*. The second required point O_{ce} must (as one of the three virtual centres of the three bodies, *a*, *c*, and *e*) be upon the line joining O_{ac} with O_{ae} which we have now the means of drawing. We have not, however, as in the last case, any other line containing it actually given by the mechanism itself, and must therefore proceed to find one. We have in our figure the point O_{bc}. This point must lie in one line with O_{ce} and O_{be}. But the latter can be easily found, for by the proof given in reference to the links *b* and *d* in the chain of Fig. 31, it must lie at the join of the axis of *a* with a line through O_{ef} normal to the axis of *b*. Drawing these two lines we get O_{be}, and joining this point to O_{bc} we have a line containing the required point O_{ce} which must therefore be at the join of this line with the one mentioned above;—its position is marked in the figure. By similar reasoning we can find the third required point O_{de}. The links *c*, *d*, and *e* being three bodies having plane motion, O_{de} must lie on the line joining O_{cd} and O_{ce}, and, for a similar reason, it must also lie upon the line joining O_{ae} and O_{ad}. Both these lines can be drawn, and their join is the required point O_{de}.

Similarly all the rest of the fifteen virtual centres belonging to the mechanism can be found, most of them in more ways than one. The only difficulty connected with the

operation is the choice of the order in which to take the points, as there are generally some which must precede others. It is not possible to lay down general rules for this, at least in any such form as to be practically useful. A little practice and experience, however, reduces this difficulty to very small dimensions.

We shall assume, in the following sections, that the virtual centre of any link in a mechanism relatively to any other can always be found, and in ordinary cases, where the methods of finding the point are those considered in this section, we shall merely give the necessary construction without special proof. We shall only give the proof in cases of some special difficulty, or where the use of higher forms of elements renders the construction in appearance—although not in reality—somewhat different from that generally adopted.

§ 13. DIRECTIONS OF MOTION IN MECHANISMS.

To find the direction in which any point of a mechanism is moving at any instant is now a very simple matter. Every point in each link is moving, relatively to any other link, at right angles to the line joining it to the virtual centre for the relative motion of the two links concerned. This line, the virtual radius of the point, can be drawn in every case, as we have seen, and we obtain at once the direction in which the point is moving by drawing a line at right angles to it. The construction is so simple that it requires no further explanation. It is illustrated in Fig. 34, where B_1, B_2 and B_3 are points of the link *b*, which is shown of general form in order to take points not lying

on its axis. The lines b_1, b_2 and b_3 show the direction in which these points are moving relatively to the link d.

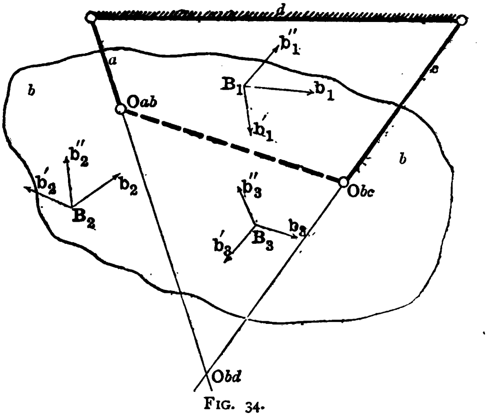


FIG. 34.

Their directions of motion relatively to a and to c are indicated by the lines b'_1, b'_2, b'_3, and b''_1, b''_2, and b''_3 respectively.

CHAPTER V.

RELATIVE VELOCITIES IN MECHANISMS.

§ 14. RELATIVE LINEAR VELOCITIES

We have so far considered motion only as change of position, entirely without reference to the time occupied by the change, that is, to the velocity of the different points of the body while moving; and we have seen that there are many kinematic problems which can be treated entirely without consideration of velocity. Connected with velocity, however, there are two distinct sets of problems which we have to examine, and one of these we can now take up. The absolute velocity of any point in a machine, as well as the changes in that velocity, depend, as we shall see presently, upon the forces acting upon the different parts of the machine. With these we have not at present anything to do. But the *relative velocities* of different points in the machine at any given instant can be determined by purely geometric considerations, so that we have already the means of dealing with them. We have seen that at each instant every body[1] in a machine or mechanism is virtually turning about some particular point, and have seen, further, how to find that point. **Every link of the machine, therefore, is simply in the condition of a wheel turning**

[1] Limiting ourselves to *plane motion;* see end of § 2.

about its axis, or a lever vibrating on its fulcrum, and this no matter how complex in appearance, or even in reality, the connection between the different parts of the machine may be. But in such a case it is obvious that the velocities of the different points must be simply proportional to their distance from the centre of rotation, that is proportional to their real, or virtual, radii or "leverage." The velocity of any one point being then known, the determination of the velocities of the others becomes a mere matter of finding the virtual centre and the distances of the various points from it. And even without knowing the *absolute* velocity of any point the same method gives us the *proportionate* velocities of all the points, quite independently of their absolute velocities. We must now look at this somewhat more in detail, especially in reference to *angular* as well as *linear* velocity.

When a body is turning about any fixed axis its motion is characterised by two conditions: (i.) the angular velocity of every point in it is equal, and (ii.) the linear velocities of its different points are proportional to their radial distances from the fixed axis, the linear velocities of points at equal distances from this axis being therefore equal. These conditions being characteristic of rotation simply, without reference to whether it occur for a short or a long time, are as entirely applicable to rotation about a virtual as about a permanent axis or centre. The difference is merely that in the former case the results obtained apply merely to one position of the body, while in the latter they apply equally to all its positions. We have seen that the motion of every link in a mechanism relatively to every other may at any instant be considered as a simple rotation about some point in that other. Hence it follows that at any instant every point in a link has the same angular velocity—that it

describes, that is, equal angles in equal times.[1] It follows also that the linear velocities of different points in any link vary in direct proportion to the virtual radii of those points. Take Fig. 35 as an example, supposing d to be fixed, and

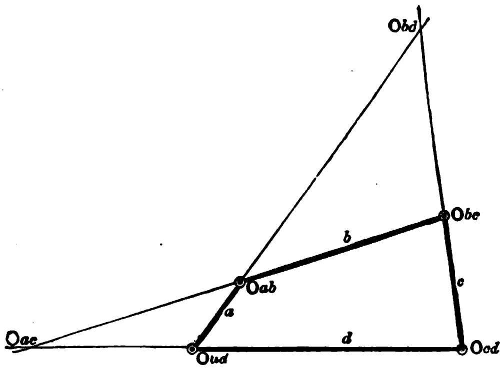


FIG. 35.

the motions of the other three links observed relatively to it. Every point in a is, at the instant, turning with the same angular velocity about O_{ad}, every point in b with the same angular velocity about O_{bd}, and every point in c with the same angular velocity about O_{cd}. Further, a point in a at any given distance from O_{ad} moves with just half the linear velocity of a point in a twice as far from O_{ad}, and with double the linear velocity of a point half its distance from O_{ad}, and similarly with the other links, whether the centres about which they are turning be permanent or virtual.

As we have seen, this makes it an extremely simple matter to find the velocity of all the points in any link if

[1] More fully that all the points *would* describe equal angles in equal times if they continued to move with the velocities which they have at the instant of observation.

only that of one point be known. Suppose, for instance, that the velocity of the point A_1 (Fig. 36) be given, to find that of A_2, both points belonging to the link *a*. Arithmetically it might be found by measuring $\overline{O_{ad}A_2}$ and $\overline{O_{ad}A_1}$, to any scale, and multiplying the given velocity by the ratio between them, *i.e.* by $\frac{\text{virt. rad. } A_2}{\text{virt. rad. } A_1}$. We shall

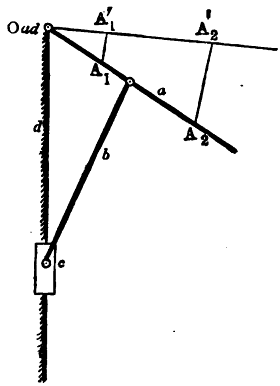


FIG. 36.

find it often more convenient, however, and it involves less measurement and no arithmetical multiplication, to solve the problem by a construction, as follows: Set off $A_1A'_1$ through the point A_1 in any convenient direction, to represent the given velocity of that point on any scale. Through O_{ad} draw a line through A'_1, and through A_2 a line parallel to $A_1A'_1$, calling the join of these lines A'_2;—the segment $A_2A'_2$ represents the velocity of A_2 on the same scale as that on which $A_1A'_1$ represents that of A_1. For the ratio $\frac{A_2A'_2}{A_1A'_1} = \frac{O_{ad}A_2}{O_{ad}A_1} = \frac{\text{virtual radius of } A_2}{\text{virtual radius of } A_1}$.

Fig. 37 shows another construction, and one often more convenient than the foregoing, for solving a similar problem. Let $B_1 B_2$ be two points of a link b, and let $B_1 B'_1$ be the known velocity of B_1, to find that of B_2. Join both points to the virtual centre of b relatively to the fixed link, viz. O_{bd}.

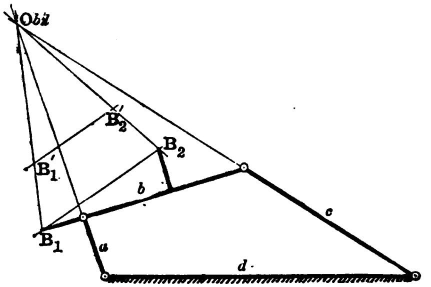


FIG. 37.

Join also B_1 and B_2, set off $B_1 B'_1$ along the radius of B_1 and draw $B'_1 B'_2$ parallel to $B_1 B_2$. Then $B_2 B'_2$ represents the linear velocity of the point B_2 on the same scale as that used in setting off $B_1 B'_1$. The proof is the same as before, simply that (by similarity of triangles)

$$\frac{B_1 B'_1}{B_2 B'_2} = \frac{O_{bd} B_1}{O_{bd} B_2} = \frac{\text{virtual radius } B_1}{\text{virtual radius } B_2}, \text{ as was required.}$$

It should be always most distinctly remembered that the bodies which are represented in our figures by straight links may be of any form whatever (see p. 67). We shall find that we have very often to do with points like B_2, Fig. 37, not lying at all on the axes of the bodies to which they belong. It should be noticed also that the line $A_1 A'_1$, &c., Fig. 36, were not set off in the direction of motion of A_1, &c., but in any direction that happened to be convenient.

We have compared the linear velocities of points of one and the same link,—but we can in just the same way compare the velocities of points in different links, or find the velocities of such points, if that of any one point be given. We do this by help of the theorem which we have already so often utilised, that the virtual centre of any link relatively to any other is a point common to both,—a point which has the same motion to whichever of the links we suppose it to belong. Let the velocity of a point A_1

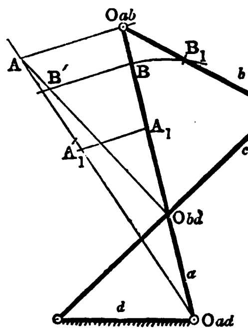


FIG. 38.

on the link a, for instance, be given;—to find from it the velocity of a point B_1 on the link b. The process is simply to find first the velocity of the common point of a and b as a point of a, and then treating it as a point in b to find from it the velocity of B_1. The necessary construction is shown in Fig. 38. $A_1A'_1$ is drawn to scale in any convenient direction for the velocity of A_1; by the former construction $O_{ab}A$ represents on the same scale the velocity of O_{ab} considered as a point of a. But this point has the same velocity as a point of b, so that by joining A to O_{bd} and

carrying the radius of B_1 round to B, as in the figure, we get BB' for the velocity of B_1, to be measured on the same scale as before.

The construction applies equally to opposite as to adjacent links. To find, for instance, the velocity of the point C_1 in c, having given the velocity of A_1 in a as before, we should proceed as in Fig. 39, finding the velocity of

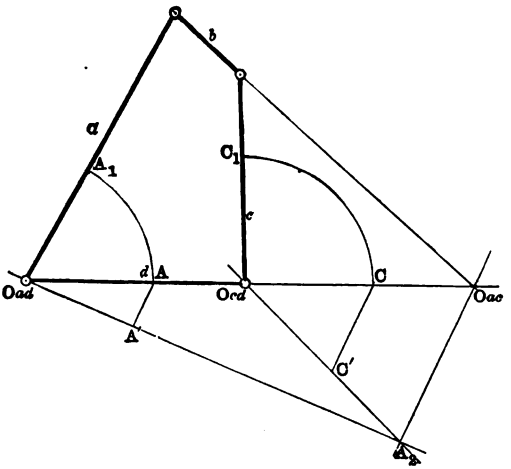


FIG. 39.

the point O_{ac} as a point in a, and then, treating it as a point of c, obtaining by its help the velocity of C_1. For convenience' sake we carry A_1 round to the line which is the axis of d, then setting off AA' as before, we obtain the line $O_{ac}A_2$ as the velocity of the point O_{ac}. Joining A_2 to O_{cd}, and carrying C_1 over to the axis of d (as we had previously done with A_1), we can at once draw CC' parallel to AA', and representing on the same scale the velocity of C_1.

Although the constructions of Figs. 37, 38, and 39 are very easy, both to work and to prove, they are not always the most convenient for practical purposes, mainly on account of the fact that it often happens that such points as O_{bd}, Fig. 37, or O_{ac}, Fig. 39, are at inconveniently great distances, often entirely beyond the limits of a drawing-board. There is never, however, any difficulty in dispensing with the actual construction of a line to such a point. Fig. 40, a very useful construction, shows how easily this

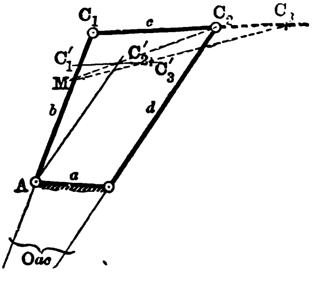


FIG. 40.

can be done. Let it be here required to find the velocity of the point C_2, that of C_1 being given. The point O_{ac} is inaccessible, but we have seen that it is sufficient for our purposes to have *any* two lines parallel to the virtual radii of C_1 and C_2, and not necessarily those radii themselves, with their (here) inaccessible join. If, therefore, we draw through O_{ab}, here the point A, a line parallel to d, we have at once such lines as we want in the most direct fashion. Setting off AC'_1 along b to represent the given velocity of C_1, and drawing $C'_1C'_2$ parallel to C_1C_2, then AC'_2 is the required velocity of C_2. For the triangles $O_{ac}C_1C_2$ and $AC'_1C'_2$ are similar, and hence $\frac{AC'_1}{AC'_2} = \frac{\text{virtual radius } C_1}{\text{virtual radius } C_2}$, which is all that is required.

If the points C_1 and C_2 were both points the directions of whose virtual radii were known, as in the figure, it would be still simpler to proceed as in Fig. 41, where $C_1C'_1$ again represents the velocity of C_1, and $C'_1C'_2$ is drawn parallel to C_1C_2. Here $C_2C'_2$ obviously gives the velocity of C_2,—it is unnecessary to go through the proof. But we

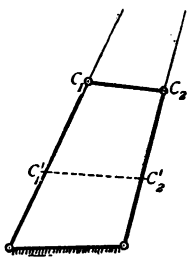


FIG. 41.

often have to deal with a point such as C_3 in Fig. 40, whose virtual radius is not a line in the mechanism, and is therefore not directly known, unless we can draw a line to the virtual centre, which is here supposed impossible. In this case we may conveniently proceed, as in Fig. 40, thus :— Find the join of the line $C_2C'_2$ with the link b, say M, and draw MC_3, which is cut by $C'_1C'_2$ in C'_3. The distance AC'_3 represents the velocity of C_3. The details of the proof may be left to the student.[1]

If in a case such as Fig. 39, the point O_{ac} be inaccessible, it can easily be dispensed with by the construction shown in Fig. 42. Here O_{cd} is joined to S, and a line drawn parallel

[1] It may be noticed that this construction gives one very easy way of drawing a line from a given point (as C_3), through an inaccessible point given as the join of two lines, b and d, on the paper. For of course a line through C_3 parallel to C'_3A would lie in the required direction.

to d in any convenient position, cutting the three lines which meet in S in the points 1, 2, and 3. The ratios of the distances between these points are exactly the same as between the three points O_{ad}, O_{cd}, and O_{ac}, which lie upon d, and it is only these ratios, not the points themselves, which we essentially require. The construction given in Fig. 39 may therefore be worked equally well by substituting 1 for O_{ad}, 2 for O_{cd}, and 3 for O_{ac}, and at the same time taking A' for A and C' for C,—in other respects working

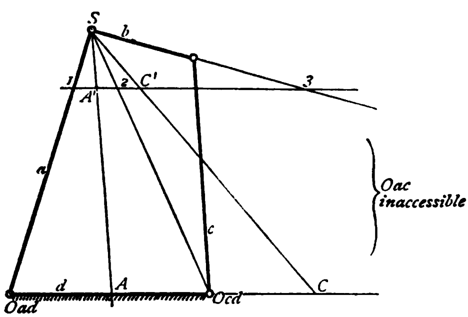


FIG. 42.

exactly as before. It can easily be seen that in effect this is nothing more than taking a different and more convenient *scale* for part of the construction. These constructions for getting over difficulties caused by inaccessible points are only examples of many that can be used, but are themselves quite generally useful, and are besides sufficient to indicate to the student the sort of procedure to be adopted in any particular case. Examples of other, more or less similar, constructions will be found further on, in connection with problems where they are required.

The whole matter which we have gone over in this section

may now be summed up. Our problem has been: Given the linear velocity v_1 of any point A of a link a in a mechanism having plane motion, to find the simultaneous linear velocity v_2 of any point C of any other link c of the same mechanism, the fixed link being (say) d. Finding first the three virtual centres O_{ac} (which we may call O), O_{ad}, and O_{cd}, we have found that

$$\frac{\text{vel } C}{\text{vel } A} = \frac{v_2}{v_1} = \frac{OO_{ad}}{AO_{ad}} \times \frac{CO_{cd}}{OO_{cd}} = \frac{OO_{ad}}{OO_{cd}} \times \frac{CO_{cd}}{AO_{ad}}.$$

Put into words this is equivalent to saying that **the velocity of C is to that of A directly as the virtual radii of those two points relatively to the fixed link, and inversely as the virtual radius in c and in a of the common point (O_{ac}) of those bodies.**

If the two points belong to the same link, the ratio $\frac{OO_{ad}}{OO_{cd}}$ goes out, and we have simply that the velocities of the two points are proportional directly to their virtual radii. Here, however, one special case requires looking at. If both the points belonged to such a body as the link c in Fig. 36, their virtual radii,—no matter what their position in the body,—would always be equal. For the virtual centre of c relatively to d is a point at infinity, the distance of which from all points in our paper must be taken to be the same. Hence if the virtual centre of a body be at infinity, *i.e.* if it have only a motion of translation, all its points are moving with equal velocities. Exactly the same thing is true in reference to the link b in Fig. 43. In this mechanism, the **parallelogram** or **double-crank**,[1] opposite links are made equal, *i.e.* $b = d$, and $a = c$. Opposite links are therefore

[1] As to important properties of this mechanism see further, §§ 54 and 55.

always parallel, and their join is always at an infinite distance;—the points O_{bd} and O_{ac} are at infinity for all possible positions of the mechanism. Whichever link, therefore, is fixed, all the points of the opposite link are moving at any instant in the same direction and with the same velocity. The difference between the case of the link *c* in Fig. 36 and that of *b* in Fig. 43 is that the virtual centre of the former

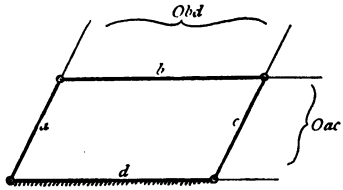


FIG. 43.

is a *permanent* centre, while that of the latter is only an *instantaneous* one. In the former case not only are all points moving in the same direction at any one instant, but this direction remains unchanged from instant to instant, whereas in Fig. 42 the direction of motion of *b* changes with every change in its position, although in any one position all its points are moving in the same direction. The difference is in essence precisely the same as that between the rotation of such links as *a* and *b* in Figs. 36 or 37. The motion of each link is at each instant a rotation about some one point. But in the case of *a* the rotation is always about the same point, in the case of *b* about a point which changes with every change in the position of the link.

§ 15. RELATIVE ANGULAR VELOCITIES.

Just as linear velocity may be expressed in different units, —as a velocity of a foot, a metre, or a mile per unit of time,—so angular velocity is a quantity measured by more than one standard.[1] The unit most commonly occurring in connection with engineering questions is *one revolution per unit of time*, the latter being generally a minute. Thus a shaft is said to have an angular velocity of 30 if it be turning at the instant at such a rate as would, if uniformly continued for one minute, cause it to make 30 complete turns in that time. To find the linear from the given angular velocity of the point in this case it is necessary simply to multiply the latter by the radius of the point and by 2π, that is, by the length of the circumference of the circle in which the point is moving. This assumes, of course, that the units of distance and of time are the same for both linear and angular velocities. For mathematical purposes the unit of angular velocity is generally taken as motion through an arc equal in length to its own radius in a unit of time. This arc subtends an angle of $\left(\frac{360}{2\pi}\right)$, or 57·3 degrees nearly, so that an angular velocity of 30 would represent, on this scale, a motion through $(30 \times 57{\cdot}3)$ degrees, or about 4·77 complete turns per unit of time. To convert angular into linear velocities on this scale the former have only to be multiplied by the radius. To convert angular velocities expressed in the former standard, therefore, to the latter, they must be divided by 2π, and *vice*

[1] The principal questions relating to linear and angular velocities are discussed in Chapter VII. What is said in the present section is not intended to do more than make clear the numerical relations of the units used as far as is necessary for the constructions given.

versâ, the time-unit being supposed the same in both cases. For general scientific purposes the second is the most convenient unit of time, but for many engineering problems the minute is preferable. For angular velocities expressed as number of revolutions, for instance, the minute is almost invariably made the time-unit.

There is obviously no more difficulty in solving problems connected with relative angular velocities than we have found in connection with relative linear velocities. It has only to be remembered, in addition to the characteristics of pure rotation already mentioned, that if two points of different bodies have the same radius, and have equal linear velocities, their angular velocities are also equal; and that otherwise (*i.e.* if the points have *unequal* linear velocities), their angular velocities are directly proportional to their linear velocities. If the two points have the same linear velocities but different radii, their angular velocities are inversely to their radii. In general, therefore, the angular velocities of two points in different bodies are proportional directly to their linear velocities and inversely to their radii. But as all points in a body must have, at each instant, the same angular velocity, we may say, even more generally, that **the angular velocities of any two bodies having plane motion are proportional directly to the linear velocities of any two of their points having the same radius, and inversely to the radii of any two of their points having the same linear velocity**, and in the general case to the ratio $\frac{\text{linear velocity}}{\text{radius}}$ for any two of their points whatever. We may put this down in symbols as follows:—calling a the linear velocity of any point A of a body α, and b that of any point B of another body β, the radii

of the points (instantaneous or permanent) being r_a and r_b respectively. Expressing angular velocities according to the second standard given on p. 95, we have—

$$\text{Ang. vel. } \alpha = \frac{a}{r_a}$$

$$\text{Ang. vel. } \beta = \frac{b}{r_b}$$

$$\text{If } r_a = r_b, \text{ therefore, } \frac{\angle \text{ vel. } \alpha}{\angle \text{ vel. } \beta} = \frac{a}{b};$$

$$\text{If } a = b, \frac{\angle \text{ vel. } \alpha}{\angle \text{ vel. } \beta} = \frac{r_b}{r_a};$$

$$\text{Or generally, } \frac{\angle \text{ vel. } \alpha}{\angle \text{ vel. } \beta} = \frac{a.\ r_b}{b.\ r_a}:$$

these three equations expressing the three conditions supposed above.

It remains only to show how, by the aid of these relations, we can find the angular velocity of any link in a mechanism having given that of any other link, or, in other words, the proportionate angular velocities of any two links. This we can always do by the method of virtual centres, generally in several ways. Taking the mechanism Fig. 44, let us compare the angular velocities of the links *a*, *b*, and *c* relatively to *d*, the angular velocity of *a* relatively to *d* being given. Comparing first *a* and *b* relatively to *d* we can proceed as follows:—*a* and *b* have a common point, the point O_{ab}; this point is therefore a point in each link which has the same linear velocity relatively to *d*. The angular velocities of *b* and *a* are therefore inversely proportional to the virtual radius in each of them of the point O_{ab}. To solve the problem

by construction draw *any line* through O_{ab} and make the segment $O_{ab}A$ equal on any scale to the given angular velocity of the link a. Then through O_{ad} draw a line parallel to the join of O_{bd} and A (which need not itself be drawn, but which is drawn for distinctness' sake in the

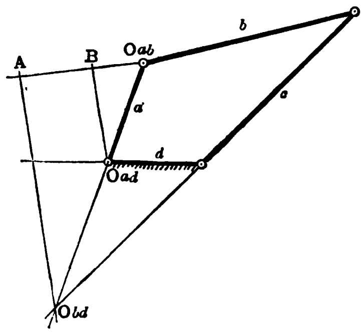


FIG. 44.

figure) and cutting the first line drawn in B. $O_{ab}B$ represents the angular velocity of the link b on the same scale as that on which $O_{ab}A$ represents that of a. The proof is simply that the triangles $O_{ab}A\ O_{bd}$ and $O_{ab}B\ O_{ad}$ are similar, and that therefore the ratio

$$\frac{O_{ab}A}{O_{ab}B} = \frac{O_{ab}O_{bd}}{O_{ab}O_{ad}} = \frac{\text{virtual radius of } O_{ab} \text{ as a point of } b}{\text{virtual radius of } O_{ab} \text{ as a point of } a}$$

exactly as required.

The construction must always have the same simplicity as in this case, for the problem always concerns the comparison of the motion of two bodies relatively to a third, and the three essential points used in the construction are the three virtual centres of these three bodies taken in pairs (here O_{ab}, O_{bd} and O_{ad}), and these points invariably, as we have seen, lie in one line.

To avoid confusion we may illustrate the other part of the problem, viz., the finding of the relative angular velocities of c and a, by another figure (Fig. 45). This case is one which has more often direct application in practice than Fig. 44. Proceeding precisely as before we take their common point O_{ac}, draw through it any line whatever on which

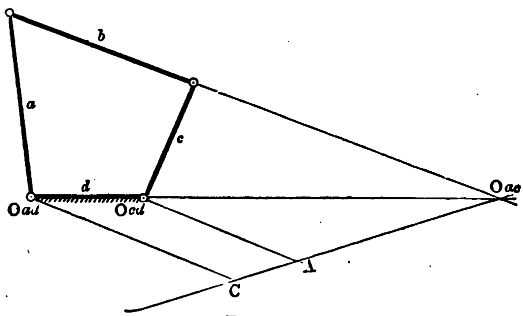


FIG. 45.

to set off a segment $O_{ac}A$ for the given angular velocity of a, and then draw $O_{ad}C$ parallel to the join of O_{cd} and A. Then $O_{ac}C$ is the angular velocity required, and

$$\frac{O_{ac}\,C}{O_{ac}\,A} = \frac{\text{angular vel. } c}{\text{angular vel. } a}.$$

Conveniently A and C may be taken on b. Alternatively we may proceed by setting off parallels through O_{ad} and O_{cd}, making $O_{cd}A$ equal to the velocity of a, and drawing $O_{ac}A$ to cut off on the other parallel the distance $O_{ad}C$, which is equal to the required angular velocity of c.

The constructions given in Figs. 44 and 45 have the advantage that they are easily proved and understood, and that in themselves they are quite simple. They have, however, some drawbacks similar to those of Figs. 37 and 39,

especially that there are many cases in which, although it is perfectly easy to find the position of such points as O_{bd} Fig. 44, or O_{ac} Fig. 45, it is practically very difficult to get at them in drawing, as they often enough lie altogether off the drawing-board. This difficulty is fortunately very easily met. All that we require is to know the *ratio* between the lengths of the virtual radii of a certain point which is common to two bodies, the ratio $\frac{O_{ab}\,O_{ad}}{O_{ab}\,O_{bd}}$ in Fig. 44, and $\frac{O_{ac}\,O_{ad}}{O_{ac}\,O_{cd}}$ in Fig. 45. This ratio we must be able to set off upon some other line than the actual line of the three centres (see also p. 92, *ante*), in such a way as to bring all

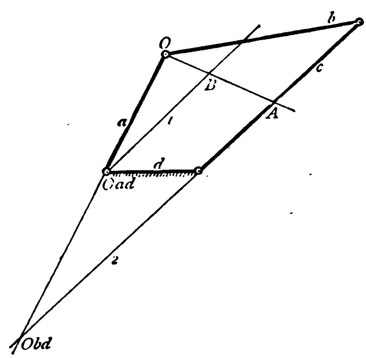


FIG. 46

the three points defining the ratio within easy reach. This can be done in an unlimited number of different ways, of which we shall point out two, one (Fig. 46) as another construction for Fig. 44, and the other (Fig. 47) for Fig. 45.

Let us call the common point, or virtual centre, of the two bodies whose relative angular velocity has to be

measured, simply O, for shortness' sake, and to distinguish it better from the virtual centres about which this point is moving in the two bodies, respectively, to which it belongs. Then we proceed thus :—(Fig. 46, corresponding to Fig. 44), —through O_{ad} and O_{bd} draw a pair of parallel lines, 1 and 2, then any line through the point O will be cut by these parallels proportionally to the virtual radii OO_{ad} and OO_{bd}. With a radius representing on any scale the known angular velocity of the link a, cut 2 in a point A, then by drawing OA we have at once OB as the required angular velocity of the link b. It will be seen at a glance that this construction does not require the direct use of the point O_{bd}, because we know that one of the links *must* pass through that point, and it is only necessary to draw the line 1 parallel to the axis of that link. To use this construction it is necessary to take the distance OA on a scale sufficiently large to allow the point A to reach the axis of c in any of its positions.

When the point O itself, and not either of the other two centres, is the inaccessible point (as in Fig. 45), a somewhat different but quite as simple method can be employed. Through the three points O, O_{ad}, and O_{cd} (Fig. 47, corresponding to Fig. 45), draw three lines meeting at one point. By taking this point at one of the opposite vertices of the figure, as S, Fig. 47, two of the lines, including that through the inaccessible point O, will be the axes of links, and only the third (the line $O_{cd}S$ in the figure) will have to be actually drawn. Then any line parallel to $O_{ad}O$ will be divided by the three lines radiating from S in the same ratio as the line of virtual centres is divided. To solve the same problem as that of Fig. 45 it is only necessary to set off $O_{cd}A$ along the line of centres to represent the known angular velocity of a, construct the parallelogram $AA'\,C'C$ as shown,

and then measure $A'C'$ or AC for the required angular velocity of *c*.

Had the angular velocity of *c* been known instead of that of *a*, it would of course have been necessary to set off

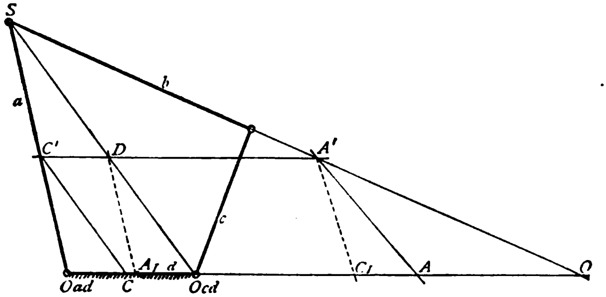


FIG. 47.

a distance $O_{ad}\,C_1$ in the first place equal to that velocity and then construct for A_1C_1, the velocity of *a*, as shown in dotted lines.

By using the second method of p. 99 (making $O_{ad}A$ = vel. *a*), we are of course already independent of O_{ac}, for the line *b*, on which we may take A and C, is always given and must always pass through O_{ac} (see Fig. 57).

§ 16. DIAGRAMS OF RELATIVE VELOCITIES.

In the last two sections we found how we could, by a simple construction, determine the linear velocity of any point in a mechanism when the linear velocity of any other point was given, or the angular velocity of any link in a mechanism when the angular velocity of any other link was given. In practice it frequently happens that it is of

interest to solve these problems for a number of different positions of the mechanism. If this has been done arithmetically the results can be represented in the form of a table; where, however, graphic methods have been used in the solution of the problem the results are most conveniently represented in a diagram, a form which on many grounds is much the more useful of the two. We propose now to work out in some detail, with the aid of illustrations drawn to scale, the methods of making diagrams of velocities as applied to several cases having considerable practical interest.

We take first a very familiar case and one of frequent occurrence. Let it be required to find the velocity with which the piston of an ordinary direct-acting engine is moving comparatively to the velocity of the crank-pin. The mechanism of the engine, the slider-crank, we have already repeatedly had before us,—it is that shown in Fig. 48.

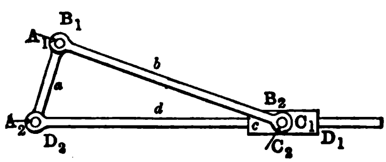


FIG. 48.

The fixed link of the mechanism becomes, in the case of the engine, the link d (see p. 67). The piston forms a part of the link c, and as the virtual centre of that link relatively to d (the point O_{cd}) is a point at infinity (p. 75), all points in it are at the same (infinitely great) distance from the virtual centre, and therefore move with the same velocity. We may therefore take *any* point in it to represent the piston, so far as its velocity goes, and for convenience' sake

we take the centre of the pair connecting the links c and b, that is the point O_{bc} Fig. 49. The crank-pin itself, considered as a solid body revolving about the point O_{ad} along with all the rest of the link a, has of course different linear velocities in its different points. What is always meant by the velocity

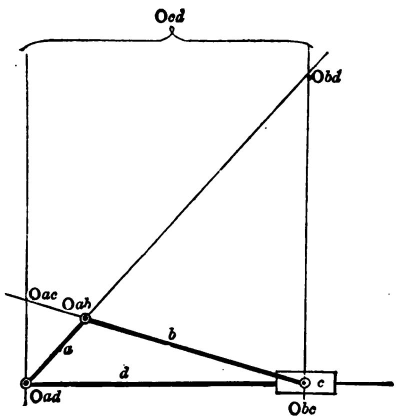


FIG. 49.

of the crank-pin is, however, the velocity of its axis, and this coincides with the centre of the pair connecting the links a and b, that is the point O_{ab}. The problem then presents itself in this form:—Given the velocity of the point O_{ab} in the plane of the fixed link d, to find for a sufficient number of positions of O_{ab} the velocity of the point O_{bc} in the same plane. One of these points is common to the links a and b, and the other to the links c and b, both of them therefore are, in all positions of the mechanism, points of the link b. Hence the problem is really nothing more than the finding of the relative velocities of two

points in the same body, as we have already done on p. 86 (Figs. 36 and 37).

Let the data in our case be as follows :—

Radius of crank (a) . . . = 1·5 feet.
Length of connecting-rod (b) = 6·0 feet.
Speed of crank 56 revolutions per minute.

This gives for the linear velocity of the crank-pin ($2\pi \times 1·5 \times 56$) = 528 feet per minute, or 8·8 feet per second. Divide the crank-circle, as in Fig. 50, into any convenient

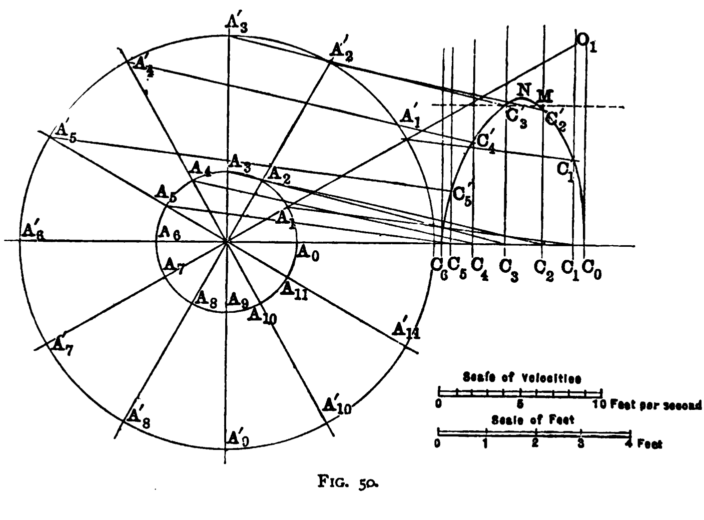


FIG. 50.

number of parts, and for each position of the crank-pin, A_1, A_2, &c., find the corresponding position of the piston C_1, C_2, &c., and the virtual centre of b relatively to d, O_1, O_2, &c. Next set off $A_1A'_1 = 8·8$ on any scale, and draw a circle about the point O_{ad} with the radius $O_{ad}A'_1$. Then lines drawn through the points A'_1, A'_2, &c., parallel to each

corresponding position of *b* will cut the corresponding virtual radii of C_1, C_2 (here parallel lines at right angles to the axis of *d*), &c., in points C'_1, C'_2, &c., such that $C_1C'_1$, $C_2C'_2$, &c., are the required velocities of the piston on the same scale as that used for setting off $A_1A'_1$. This construction has already been proved in connection with Fig. 37. A curve joining all the points C'_1, C'_2 &c., gives by its ordinates the velocity of the piston at any required point of its travel.

If it be required rather to represent the velocity of the piston at each position of the crank instead of at each of its own positions, then what is called a *polar* diagram, like that shown in Fig. 51, can be used. Here a circle is drawn

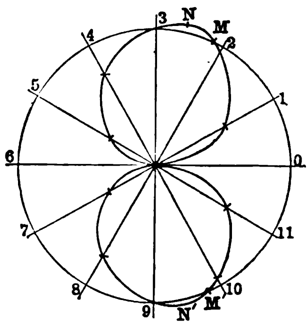


FIG. 51.

with radius = 8·8 on the same scale as before, the different positions of the crank are marked on it, and upon each the corresponding distance $C_1C'_1$, $C_2C'_2$, &c., is set off.

The actual changes of velocity in this case are worth noticing. At the beginning of the stroke the velocity of the piston is of course nothing. This comes out in the construction by the coincidence of the virtual centre with

the point C_0. As the piston moves on its velocity increases. At first the angle at C (as $A_1C_1O_1$) is greater than that at A (as $C_1A_1O_1$) so that OA is greater than OC, and therefore the velocity of the piston *less* than that of the crank-pin. At some point, however, the angle at A must become a right-angle (when the axis of the connecting-rod b is tangential to the crank-circle), and then OC must be the hypotenuse of a right-angled triangle, and therefore *greater* than OA, so that the velocity of the piston must be then *greater* than that of the crank-pin. Before this point is reached there must, therefore, be some position in which the velocities of the piston and crank-pin are equal, and it is obvious that this position will be that for which the triangle, AOC, is isosceles, and $OA = OC$. When the crank-pin is in its central position, A_3, and at right-angles to the direction of the piston's motion, OA and OC are again equal, the two virtual radii being parallel, and the point O at infinity. Here again, therefore, the velocity of the piston is equal to that of the crank-pin. After this, until the end of the stroke, OA is always greater than OC, and the velocity of the crank-pin therefore greater than that of the piston, and at A_6 the latter again becomes = 0, *i.e.* the piston is for the instant stationary. The same changes of relative velocity occur, in reversed order, as the crank makes its second half revolution from A_6 back to A_0—the lines for these are not shown in the diagram.

The positions of the important points noticed are readily seen in Fig. 51; and also in Fig. 50 if a line be drawn parallel to the axis of d and at a distance from it $= A_1A'_1$, or 8·8. At O and at 6 (or C_0 and C_6) the ordinate of the piston speed-curve is = 0. At M and at 3 (and also at 9 and at M') it is equal to the distance which represents the crank-pin velocity, while at N it exceeds that distance.

The actual maximum velocity of the piston would in this case be about 9·3 feet per second, or 5·7 per cent. greater than that of the crank-pin.

It will be found an interesting exercise to draw diagrams of this kind for different lengths of connecting-rod, and note how considerably the shortening of this rod increases the maximum velocity of the piston. It will easily be seen also that while for all lengths of the rod the point 3 (the second point at which crank-pin and piston velocities are equal) remains in the same position, the points *M* and *N* move nearer and nearer to it as the rod is lengthened. If the rod could be made infinitely long these three points would coincide, the maximum velocity of the piston would be equal to the crank-pin velocity, and the position of maximum velocity would occur when the crank is in its mid-position. We have already pointed out that the mechanism of Fig. 52, some properties of which were examined in

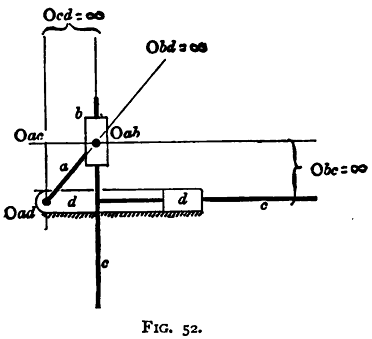


FIG. 52.

§ 12, is constructively equivalent to the one with an infinitely long connecting-rod. It will be worth while to work out for this mechanism a diagram similar to that just worked out for the slider-crank. This is done in Fig. 53, which is

drawn to the same scales as Fig. 50, in which also the same data as to speed are assumed, and the same length of crank *a*. It is more convenient in this case, for reasons

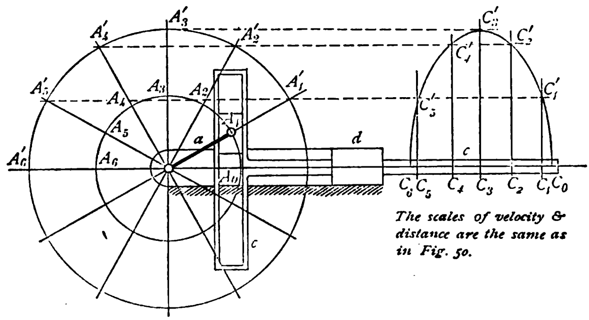


FIG. 53.

which can easily be seen, to set off the crank-pin velocity (8·8 feet per second) from the point O_{ad} instead of, as in Fig. 50, from the point O_{ab}. Then marking the positions A_1, A_2, A_3, &c., as before, and taking any convenient point upon c to represent the piston, the piston velocities, $C_1C'_1$, $C_2C'_2$, &c., are at once found by drawing the lines $A'_1C'_1$, $A'_2C'_2$, &c., parallel to the connecting-rod, *i.e.* parallel always to the direction of motion of the piston, for the infinitely long connecting-rod moves (as we saw on p. 77) always parallel to itself. It will be seen at once that the curve $C'_1C'_2C'_3$ . . . . . is a semi-ellipse of a height equal to the radius of the velocity circle,—that the maximum velocity of the piston is *equal to* the maximum velocity of the crank, and that the maximum velocity is reached by the piston just when the crank is at mid-stroke—these being precisely the characteristics which we pointed out a few lines back as theoretically belonging to

a mechanism with an infinitely long connecting-rod. In Fig. 54 is shown a polar diagram for this case corresponding to Fig. 51. Here again there is a great simplification, the two curves are simply circles having

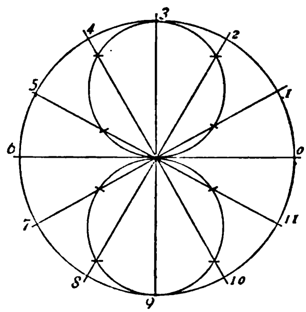


FIG. 54.

diameters equal to the length which stands for the crank-pin velocity.

It may be noted here that sometimes one wishes simply to find out the relative velocities of the piston at different periods of its stroke without reference to any particular crank-pin velocity. In this case it is convenient either to assume the crank-pin velocity = (say) 10 on any convenient scale, or (sometimes) to let the radius of the crank itself stand for its velocity. In the last case the diagram of Fig. 54 becomes very similar in appearance to the well-known Zeuner valve diagram, although its interpretation is very different.

As a last example of the construction of diagrams of velocities we shall take a chain similar to that already shown in Fig. 25 and dealt with in Figs. 40 and 44, &c., but with

different proportions. Take the length of the four links as follows :—

$$\begin{aligned} a &= 14 \text{ inches} \\ b &= 32 \quad ,, \\ c &= 19 \quad ,, \\ d &= 34 \quad ,, \end{aligned}$$

and make the link *a* the fixed link. Suppose *b* to turn with a uniform angular velocity of 48 revolutions per

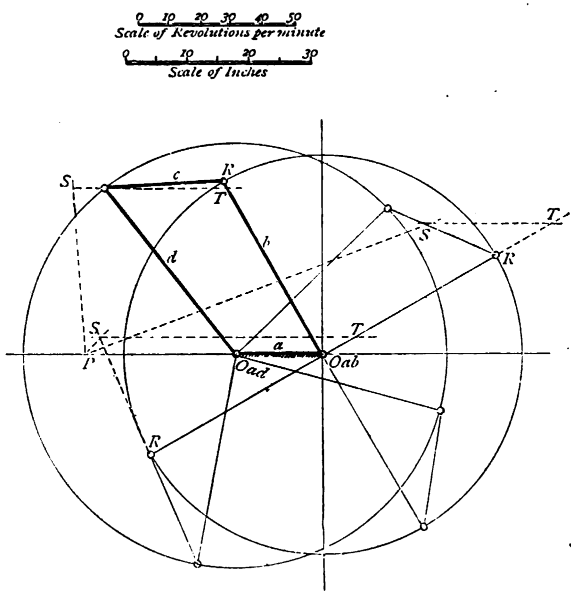


FIG. 55.

minute, and let our problem be to find the angular velocity of *d* at any number of different positions of *b* (Fig. 55). In a case of this kind it is most convenient to use some

such abbreviated construction as was given in Fig. 47. First set off any required number of positions of the link *b*, and for each construct the corresponding positions of *d* (a few of these positions are shown in the diagram). From the point O_{ad} (*i.e.* the virtual centre relatively to the fixed link of the link *d*, whose velocity has to be found) set off upon the axis of *a* a distance $O_{ad}P$, standing for 48, the angular

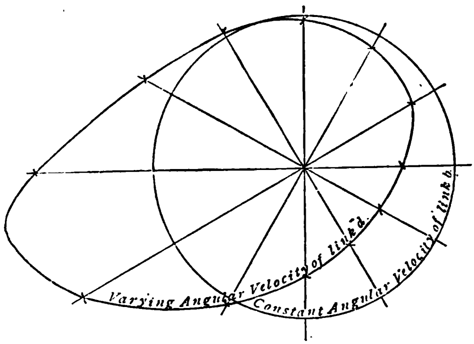


FIG. 56.

velocity of *b*, upon any scale. Then for each position of the mechanism draw a line through *P*, parallel to the line (which need not be itself drawn) joining O_{ad} and *R*, until it cuts the axis of *c* in a point *S*. The distance *ST*, from *S* to the axis of *b* (measured parallel to the line of the three centres, here the axis of *a*) is the required angular velocity of *d*. A diagram may be conveniently made, as in Fig. 56, by drawing a circle with a radius = 48 to stand for the constant angular velocity of *b*, marking on it the positions of *b* used in Fig. 55, and then setting off on each radius the corresponding value of *ST*, the angular velocity of *d* for the given position of *b*.

Such constructions as those given in Figs. 47 or 55 are very general, allowing the velocities to be represented by any distances whatever, *i.e.* drawn on any scale whatever. But by using particular distances and scales the construction may often be greatly simplified. Thus if we take the length of the link *d* in Fig. 55 to represent the angular velocity of *b*, we have only to draw through O_{ab} a line parallel to *d*, and cutting *c*, to obtain at once the angular velocity of *d* on the same scale. By comparing this construction (Fig. 57) with

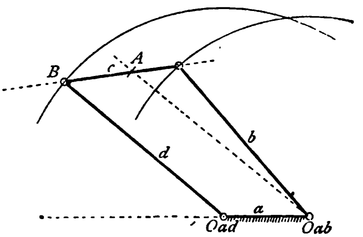


FIG. 57.

that of Fig. 45 it will be seen that *BA* is a line passing through the (inaccessible) centre O_{bd}, and is in exactly the same position as the axis of *b* or the line *CA* in that figure. If this construction, which is the simplest of all possible constructions for this case, be used, the circle in Fig. 56 has, of course, a radius equal to that of the link *d*.

We should have obtained the same result in Fig. 45 if we had not only taken *A* and *C* on the axis of *b*, as there mentioned,[1] but had set off $O_{cd}A$ on such a scale as to be equal to *c*. Whether the greater practical convenience lies in the saving of construction lines, or in the use of an even scale of velocities, may be left to the draughtsman to decide in each particular case.

[1] See also end of § 15.

It must be noted that the *mean* angular velocity of *b* and *d* must be equal, for each one takes the same time to make one whole revolution, but if *b* have a *uniform* velocity within the revolution, then *d* has the varying velocity which has been drawn in Fig. 56. In the case supposed, the velocity of *d* would vary within each revolution from the rate of $27\frac{1}{2}$ revolutions per minute to that of 97 revolutions per minute, the actual mean rate being 48 revolutions per minute.

If we had taken the chain of Figs. 32 or 53, and fixed the link *a*, we should have obtained a mechanism in which, as in that of Fig. 55, the links *b* and *d* would both revolve, the one driving the other through the link *c*. This mechanism is that generally known as an "Oldham coupling." It is interesting to try with it the constructions of Figs. 45 or 55 for finding the relative angular velocities of the links *b* and *d*. It will be found that the latter construction gives no result, too many of the points required being at an infinite distance, but the former shows very clearly the leading characteristic of the mechanism, that the angular velocity ratio transmitted between the shafts (the links *b* and *d*) is constant, and is equal to *unity*.

An important analogue of this mechanism, the "universal joint," will be examined in detail in § 64.

A possible misunderstanding may be guarded against before we leave this part of our subject. Let *b* and *c* be any links of a mechanism of which *a* is a third link. We have seen how to find the ratio $\frac{\angle \text{ r. vel. } b}{\angle \text{ r. vel. } c}$ when both these velocities were measured relatively to *a*. This ratio must not be confused with the angular velocity of *b* relatively to *c*, which is of course an entirely different quantity. If the angular velocities of *b* and *c* relatively to *a* be β and γ respectively,

then in order to find the angular velocities of these two links relatively to each other we have only to proceed as in § 3—namely, give to both a velocity equal and opposite to that of one of them. Thus if we give to both an angular velocity of $-\gamma$, we bring c to rest and find the angular velocity of b relatively to it to be $\beta - \gamma$. Or if we give both an angular velocity of $-\beta$, we bring b to rest, and find the angular velocity of c relatively to it to be $\gamma - \beta$, which is the same magnitude as before, reversed only in sense.[1] In general, if v_1 and v_2 be the angular velocities of any two bodies relatively to a third, the velocity of each relatively to the other is $v_1 \pm v_2$, where the *positive* sign is to be used if v_1 and v_2 have opposite senses, and the *negative* sign if they have the same sense. Thus in Fig. 56 distances measured radially between the two curves represent the angular velocity of b relatively to d on the same scale as that on which the radii of the curves represent the angular velocities of b and d respectively relatively to a.

[1] § 3, p. 27.

CHAPTER VI.

MECHANISMS NOT LINKWORK.

§ 17. SPUR-WHEEL TRAINS.

THERE are comparatively few mechanisms in general use in which the *surface* contact of such pairs of elements as the pin and eye is replaced by the *line* contact[1] of higher pairs. In toothed-wheel gearing, however, we have one type of mechanism with higher pairs which is very familiar, and which is important enough to require some detailed consideration. There are many forms of toothed gearing, but here we shall consider only those which have plane motion, and which are usually distinguished by the name of **spur gearing**, or spur-wheel trains. It will be found that the methods already employed in the examination of linkworks can be employed here also with equal ease and with equally practical results.

The commonest example of a spur-wheel train is shown in Fig. 58. It is a chain containing three links only, of which one, *a*, is a frame, while the others, *b* and *c*, are wheels. Between *a* and *b* and between *a* and *c* are ordinary turning pairs; between *b* and *c* the connection is by means of the wheel-teeth, which form a *higher* pair, having line-contact only.

[1] See § 10, p. 57.

We shall first of all find the virtual centres for this mechanism. These are only three in number, O_{ab}, O_{bc}, O_{ac}, and therefore must, as we know, all lie upon the same straight line. This line must clearly be the axis of the

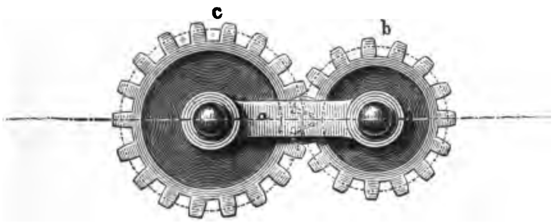


FIG. 58.

link a, for we see at once that O_{ab} and O_{ac} lie upon that axis. It remains therefore only to find the position of O_{bc} upon that line. This cannot be determined by any direct construction, but we can find it by very simple reasoning of a quite general kind, which can afterwards be applied to the special case of spur gearing. Spur-wheels are bodies which revolve (or are intended to revolve) about fixed axes with a fixed velocity ratio. Let b and c (Fig. 59) be any two such bodies, and O_{ab}, O_{ac} their fixed axes. The problem is then to find the virtual centre O_{bc}, which we already know to lie upon the line O_{ab} O_{ac}. O_{bc} is a point common to b and to c, and must therefore have the same linear velocity in each. Its virtual radii in b and in c must therefore be proportional inversely to the (known) angular velocities of those bodies. But these virtual radii are simply its distances (measured along the given line) from O_{ab} and O_{ac} so that O_{bc} must be a point whose distances from O_{ab} and O_{ac} are inversely proportional to the angular velocities of b and c, or to the number of revolutions made

by each in a given time. Let this known ratio of $\frac{\angle \text{r. vel. } b}{\angle \text{r. vel. } c}$ be equal to $\frac{n}{m}$, then to fix the point O_{bc} it is only necessary

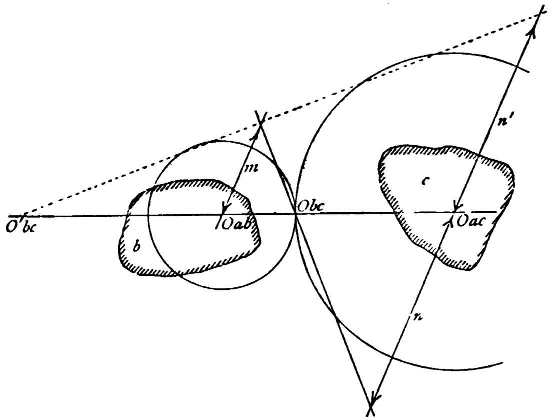


FIG. 59

to draw two parallels through O_{ab} and O_{ac} to set off n from O_{ac} on the one, and m from O_{ab} on the other, on opposite sides of the centre line, and to join the end points of the segments m and n, as in Fig. 59.[1] By similar triangles it is then evident that

$$\frac{O_{ab}\,O_{bc}}{O_{bc}\,O_{ac}} = \frac{m}{n} = \frac{\angle \text{ r. vel. } c}{\angle \text{ r. vel. } b},$$

[1] The quantities n and m are to be set off on opposite sides of the centre line only if the two bodies b and c are required to turn in *opposite* senses. If they are to turn in the *same* sense n and m must be set off on the *same* side of the axis, and the point O_{bc} will lie *outside* the other two points, as shown by the dotted line, instead of between them, which corresponds to one of the wheels being *annular*. It can easily be seen that the two points O_{bc} which can thus be found for every ratio $\frac{n}{m}$, must be harmonic conjugates with respect to the two fixed points.

that is, that O_{bc} is a point on the line of centres whose distances from O_{ab} and O_{ac} are inversely proportional to the angular velocities of b and c. O_{bc} is therefore the point which we require, the virtual centre of b relatively to c. As the bodies b and c rotate different points in each become in turn their common point, or virtual centre. But as the virtual centre must always occupy the same position between O_{ab} and O_{ac} the different points in b and in c which successively become O_{bc} must be such points as can, during the rotation of the bodies, occupy that position. The locus of such points in b is of course a circle with centre O_{ab} and radius $O_{ab}O_{bc}$ and in C a circle with centre O_{ac} and radius $O_{bc}O_{ac}$. These two circles (which are shown in the figure) are the centrodes for the relative motions of b and c. As these bodies rotate with the given angular velocity ratio, the centrodes roll upon one another. They correspond exactly to what are technically called the **pitch circles** of the wheels.

We have already seen that the relative (plane) motion of any two bodies is always conditioned by the rolling on each other of two curves,—centrodes, or loci of virtual centres (§ 9),—one supposed fixed to each of the bodies. So long as the rolling centrodes remain unaltered, the motion cannot be changed, and conversely so long as the motion remains unchanged the same centrodes must roll on each other. Given the motion, pictorially or otherwise, we can find the centrodes,—given the centrodes similarly, and the motion is equally determinate. In general, however, the form of the centrodes is very complex, and it is possible only in exceptional cases to make any use of them.[1] Here

[1] The best examples of the direct use of any part of them in ordinary mechanisms are perhaps the exceedingly ingenious constrainments of certain mechanisms having change-points devised by Reuleaux. See *Kinematics of Machinery*, figures 155, 159, etc.

it is otherwise, the centrodes are circles whose centres and radii are determinate in the simplest possible manner, and their use is both easy and convenient.

We desire to make two bodies *b* and *c*—we may call them wheels—revolve about fixed axes with a constant velocity ratio. We find that this motion is conditioned absolutely by the rolling on each other of two circles of known radii, whose centres lie in the given axes, and which touch in O_{bc}. Dealing with bodies instead of plane figures, the circles of course become circular cylinders, the point O_{bc} becomes a line parallel to the axes, and we see at once that in order to communicate the required rotation between the bodies, it would be sufficient to shape them as cylinders touching along a line through O_{bc} if only it could be practically insured that the surfaces of the cylinders should not slip upon each other. For then the linear velocity of all points in the surfaces of the cylinders would be equal, and the angular velocities of the cylinders would be inversely as their radii. On account of the difficulty of insuring in practice that two such cylinders shall turn absolutely without slipping, their surfaces are generally provided with teeth of sufficient size and strength to compel the one wheel to turn when the other is moved. These teeth form elements of higher pairs connecting the two bodies; if they are to communicate the same relative motion as before they must be formed so as to correspond to the same centrodes as before. And as these curves are so simple it is possible to use them directly in finding the right shape for the teeth, as we shall see in the next section.

By the use of teeth of almost any practicable shape it is insured at once that at least the *average* relative velocities of the wheels shall have their intended value. But if the wheel-teeth are not of the proper form, not only may

a great deal of loss by friction occur (which will be considered further on), but also the relative velocities of the two wheels will be continually varying between a maximum above and a minimum below the average ratio. The mere changes of velocity are not in themselves generally large or inconvenient, but indirectly they are the cause of the greater part of the noise which so often accompanies the working of toothed gearing, and therefore of the wear of which noise is the sure indication. The correct formation of the teeth of wheels is not, therefore (as is sometimes supposed), a matter of purely theoretical interest, but one which has a very direct practical importance. It has too often been neglected in the manufacture of toothed wheels, under the impression that it was not only a very complex affair, but also that it had no practical value, and as a result teeth have been used whose forms are only rough approximations to any accurate shape, and the working of the gear has been intolerably noisy and disagreeable. In reality it is just as easy to draw the right shape for wheel-teeth as to construct any of the ordinary approximations[1] to it. To insure, however, that the teeth are actually of the right shape (when it has been found), the very greatest care is required in the moulding of the wheels. In many cases it has been found worth while to machine the surfaces of the teeth, in spite of the expense of the process and the practical drawback that it removes the hard "skin" of the metal, which, both for strength and for wear, it would be preferable to preserve.

[1] The best approximation (a very good one), is, no doubt, that proposed by Professor Unwin, *Elements of Machine Design*, p. 259 (Fifth Edition).

§ 18. WHEEL-TEETH.

THE essential condition to be fulfilled by the forms or *profiles* of wheel-teeth is, that **at whatever point there may be contact between the teeth of the driving and the driven wheels, the velocity-ratio of the two wheels must be the same.** But we have seen that for any given velocity-ratio there is some fixed virtual centre for the two wheels. We might, therefore, express the condition by saying: **at whatever point there may be contact between the teeth, their virtual centre must remain unchanged.**

The only motion which one tooth can have relatively to another, at the line where the two are in contact, is that of *sliding*. The direction of motion of this line in the one tooth relatively to the coincident line in the other must therefore be the direction in which the one can slide on the other, that is, the direction of the tangent plane to the two surfaces, or if we consider the profiles of the teeth only, we may say the direction of the tangent to the two profiles at their point of contact. The virtual centre must be in a line normal to this tangent (see § 7), and as the virtual centre is always a known point (the intended velocity-ratio of the wheels being known), the tangent to the profiles for any given position of the point of contact can always be drawn. In Fig. 60, for instance, the curves b_1 and c_1 might form part of the profiles of teeth for wheels b and c, whose virtual centre is at O_{bc}. For the tangent t at the point of contact O is at right angles to the virtual radius of that point, the line $O O_{bc}$. But if the profiles were formed as at b_2 and c_2, although b_2 might still drive c_2, or *vice versâ*, these curves could not form part of the required profiles, because the normal at O, the line

which is the virtual radius for the relative motion of the two profiles, does not pass through the required virtual centre O_{bc} but cuts the line of centres in some different point, a point corresponding to totally different centrodes, and therefore to a totally different velocity-ratio from the one which it is desired to transmit.

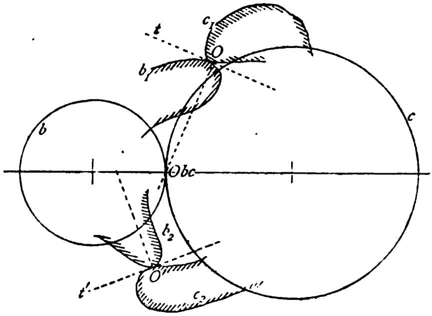


FIG. 60.

It is easy to draw a curve which could not possibly be used as the profile for a tooth of a given wheel, but given any curve which *could* be so used, there is no great difficulty in finding the form which the profile working with it must have *in order to fulfil the necessary conditions.*[1] It is not consistent with our purpose here to describe the methods for doing this, or to take up other special cases connected with the form of teeth, although they are both numerous and interesting. In the immense majority of cases which occur in practice, the profiles of wheel-teeth are made more or less nearly to cycloidal arcs, because these curves fulfil

[1] This problem, as well as a number of other special cases relating to the form of wheel-teeth, have been treated at length and with great clearness, by Reuleaux, both in his *Constructeur* and in the *Kinematics of Machinery*, pp. 146—164, &c. See also Unwin, chap. ix.

exactly the conditions which we have laid down as being essential to the proper working of wheel-teeth. It will be sufficient for our purposes here to prove that this is the case. Cycloidal curves are not used because of any intrinsic value which that class of curve possesses, but simply because they are the most easily constructed of all the curves which fulfil the desired conditions and give at the same time teeth of a shape having the necessary strength and being in other respects practically convenient.

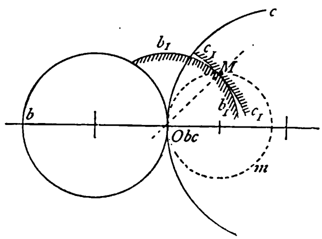


FIG 61.

In Fig. 61 let b and c be the centrodes or pitch-circles of a pair of wheels in contact always at O_{bc} and let m be a third circle, which is to be used as a "describing circle" in drawing the cycloidal profiles, and which is in contact with both at the same point, the point M being the "describing point." When one curve rolls upon another the virtual centre for the relative motion of the two curves is always their point of contact. For as the motion of rolling excludes *slipping* by definition, the curves must be stationary relatively to each other at the point of contact, and such stationariness, as we have seen (p. 44) can only occur at *one* point of two bodies (unless the bodies are fixed to each other altogether), and that point is

their virtual centre. When m therefore rolls in or upon b or c its virtual centre relatively to either is its point of contact with it. The point M then must describe curves which have the property that their direction at any point is normal to the line joining that point to the point which is O_{bm} or O_{cm} for the time being, that is to the point which is the point of contact of b and m or of c and m. If now b and c be supposed to turn about their centres and roll upon one another at the same time, so as to be always in contact at O_{bc} and if further m roll at the same time upon both curves, the points O_{bm} and O_{cm} will both coincide always with O_{bc} and the point M will describe simultaneously a curve b_1 on b and a curve c_1 on c. If these curves b_1 and c_1 be now used as profiles of teeth, the one driving the other, they will remain in contact (for they have been described simultaneously) and their common normal at the point of contact will always pass through O_{bc}. They will fulfil, in one word, precisely the condition that the teeth were required to fulfil, and constrain a motion whose velocity-ratio is constant and is of the required value, and whose centrodes are the given pitch circles. These curves are however precisely the epicycloid and hypocycloid drawn by the same describing circle upon the pitch circles which are generally chosen for the ideal forms of wheel-teeth, and here we have the real cause and justification of that choice.

It is obvious that the proof just given is independent of the size of the describing circle m. If, however, its diameter be made more than (or even as much as) half that of the smallest wheel for which it is to be used, the teeth of that wheel, although kinematically correct, will have a form which is narrower at the bottom than at the middle, and which is consequently deficient in strength. The *maximum* diameter of a describing circle is generally fixed, on these

grounds, at half the diameter of the smallest pitch circle for which it is to be used.

It is necessary to use the same describing circle for parts of teeth which are to work together, but it is not necessary to draw the whole tooth with the same describing circle. In Fig. 61, for instance, the inside of *b*, and the outside of *c*, might be drawn with a circle of different diameter to *m*. On practical grounds, however, it is advisable to use the same describing circle, not only for the whole teeth of any one pair of wheels, but for as many different wheels as possible. The reason for this is that all wheels whose teeth have been drawn with the same describing circle will work correctly with each other, and this is often a great convenience in practical work. Such wheels may be called **set wheels.**

The portion of a tooth outside the pitch circle is called its **point,** the portion inside the pitch circle its **root.** The profile of the point is called the **face** of the tooth and the profile of the root the **flank.** It is necessary that we should now consider the relative motions of these profiles on each other. Suppose that a pencil had been attached to any point of the describing circle *m* (Fig. 61) while *b*, *c*, and *m* were simultaneously rolling together. Relatively to *b* and *c* the pencil would have described the cycloidal curves b_1 and c_1, and the particular position of the pencil (as *M*) at any instant would mark the point of contact of those curves at the instant, because, by hypothesis, the pencil point lies simultaneously on both curves. But relatively to the plane of the paper, the pencil point will simply describe the circle *m*. **The path of the point of contact of a pair of cycloidal teeth is therefore an arc of the circle with which their profiles have been described.**

Fig. 62 shows a pair of such teeth in their first and last contact. The path of the point of contact is the double arc A_1OA_2. The wheels are lettered *a* and *b*, and the frame (not drawn) forms the link *c*. The root of a tooth of *a* drives in the first instance the point of a tooth of *b*, and this

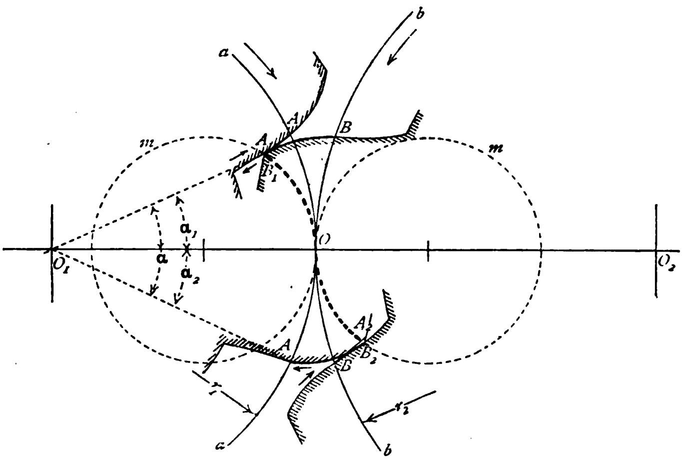


FIG. 62.

continues throughout the angle α_1 (called the **angle of approach**) until the pitch points *A* and *B* of the teeth fall together at *O*. Relatively to each other the two surfaces are moving in the sense indicated by the small arrows. If at *O* the point *B* had slid into contact with the same point A_1 as came first into contact with B_1, the amount of sliding of *b* relatively to *a* would have been simply B_1B. But as *B* eventually comes into contact with *A* and not with A_1, the real amount of sliding is only $B_1B - A_1A$. The latter quantity (A_1A) may be called the **working length** of the flank. Similarly during α_2 the **angle of recess**, the teeth

rub on each other until finally A_2 is in contact with B_2, and the amount of sliding is $AA_2 - BB_2$. The total amount of sliding of one pair of teeth, then, during their whole period of contact (corresponding to the arc $\alpha = \alpha_1 + \alpha_2$) is

$$B_1B - A_1A + AA_2 - BB_2$$
$$= B_1B - BB_2 + AA_2 \quad A_1A = s$$

or the sum of the lengths (length of face – working length of flank) for the two teeth.[1]

Every point on the pitch circle of the wheels has, during the time the pair of teeth were in contact, moved through a distance $r_1\alpha$ (the angle α being in circular measure). We have then the ratio

$$\frac{\text{Mean velocity of rubbing of teeth}}{\text{Velocity of pitch circle}} = \frac{s}{r_1\alpha}.$$

The velocity of the pitch circle is $2\pi r_1 n$ (n being the number of revolutions of a per minute). The mean velocity with which the teeth slide on one another is therefore $s.n.\frac{2\pi}{\alpha}$, or (if we write V for the pitch circle velocity) $= V\frac{s}{r_1\alpha}$.

It is possible to obtain this velocity in a quite different form. If we write a_b and a_c for the angular velocity of a relatively to b and c respectively, and b_c for the angular velocity of b relatively to c, then we know (§ 16, p. 115) that

$$a_b = a_c + b_c$$

(the sign is positive because a and b turn in opposite senses relatively to c). The point O is the virtual centre of a

[1] The first and last points of contact for any given teeth can be found at once as the points where circles drawn about O_2 and O_1, with radii equal to O_2B_1 and O_1A_2 respectively, cut the describing circles m.

relatively to b. The linear velocity of any point in a relatively to b is equal to a_b multiplied by the distance of the point from O. The velocity of A_1 therefore, which is the velocity of sliding at the first point of contact, is

$$a_b \times OA_1 = (a_c + b_c)\, OA_1.$$

But $a_c = \frac{V}{r_1}$ and $b_c = \frac{V}{r_2}$. Substituting these values, and taking the mean velocity as half the initial velocity, which is very closely approximate, we get for the mean velocity of sliding

$$V\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\frac{OA_1}{2}.$$

In order that two pairs of teeth may always be in contact it is necessary that the *arc* OA_1 should be at least equal to the pitch. Taking the arc and the chord as equal, it is very usual to substitute p, the pitch of the teeth, for OA_1 in the formula just given. The result is of course more exact if $\frac{ra}{2}$ be substituted for OA_1 (assuming a_1 and a_2 to be equal).

It will be noticed that the sliding of the teeth upon one another is not by any means exactly the same thing as the sliding which occurs in the surfaces of lower pairs. Here at any one instant the two lines of the tooth surfaces which are coincident slide upon one another in a direction normal to their virtual radii. (If a be fixed, for instance, and b be made to move, B_1 slides upon A_1 in a direction normal to OB_1.) But from instant to instant the position of the virtual centre changes, and therefore the direction of the virtual radius. The *direction of sliding* is therefore different for each position of the surfaces. With the lower pairs the sliding at any point remains constant in direction throughout

the motion of the pair, because the virtual centre of the pair is permanent. The difference between the sliding in higher and in lower pairs is thus closely analogous to the difference between general rotation about a series of virtual centres, and permanent rotation about a fixed axis.

This general case of sliding between two surfaces, for which no special name has been proposed, is the one which occurs not only in wheel teeth, but in all higher pairing, including cams (§ 22) and the "reduced" chains of § 53. The closed higher pairs of Reuleaux[1] form excellent illustrations of it. Outside the lower pairs pure sliding only occurs in certain very special cases, of which one or two are mentioned in § 57. The same relative change of position as that corresponding to any finite amount of general sliding can be produced also by rolling and pure sliding added or superposed.

The smaller the angle a (Fig. 62) can be made, the more nearly will the path of contact approach a straight line at

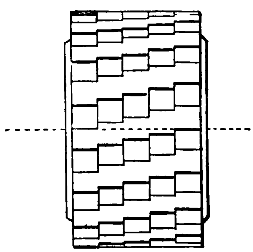


FIG. 63.

right angles to the line of centres, the less will be the obliquity of the pressure between the teeth (the effects of which we shall examine in § 76), and the less will be the extent of the

[1] *Kinematics of Machinery*, §§ 21 to 29.

sliding of the teeth. To attain these results without reducing the pitch (which would weaken the teeth) Dr. Hooke long ago proposed making wheels with stepped teeth (Fig. 63). In recent practice the same result has been attained by making the steps continuous, so that the outlines of the teeth become portions of screws. To prevent endlong pressure on the shaft the screws are made double, as shown in Fig. 64. With the improved appliances now in use, it is found

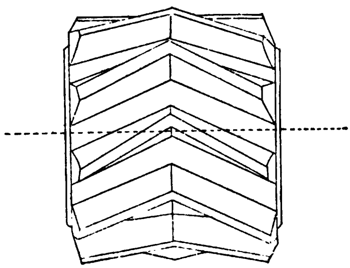


FIG. 64.

possible to make these teeth very accurately, in spite of their complexity of form, and wheels so made work exceedingly smoothly. Screw wheels proper, with non-parallel axes, are dealt with in § 69.

§ 19. COMPOUND SPUR GEARING.

THE mechanism with three links which we have been considering in the last two sections is the simplest form which spur-wheel gearing can take. Very frequently chains containing toothed wheels are *compound*, which it will be remembered (see p. 68) means that one or more links carry more than two elements. Such a chain is shown in Fig. 65,

where the frame *a* has three elements, and the wheel *d* also the same number, being paired with *a*, *b*, and *c*.[1] In this train, by its construction, the points O_{bd} of *b* and O_{cd} of *c* have the same linear velocity in *opposite* senses, whereas in the trains just considered the same points had the same linear velocity in *the same* sense. So far as regards the relative motion of *b* and *c*, therefore, the insertion of the third wheel *d* between them has made no difference except in sense. Formerly they turned in opposite senses, now they turn in the same sense, their velocity-ratio being unchanged. The *size* of *d* does not affect the motions of *b* and *c* in any way whatever. Such a wheel as *d*, which merely affects the sense of motion in a train without altering the relative velocities of the first and last wheels, is called an **idle wheel.** Any number of idle wheels may be inserted between the first and last wheel of a spur train; if the number of idle wheels be *odd* the first and last wheels will turn in the same sense, if *even* in the opposite sense. No other change is made.

If $\frac{n}{m}$ be the angular velocity-ratio of *b* and *c*, which is called the velocity-ratio of the train, then to find O_{bc} we must set off these quantities on parallels, as in Fig. 59, on the *same* side of the axis (see note p. 118) instead of on opposite sides, and join their end-points as before. As $\frac{\text{rad. } c}{\text{rad. } b}$ is equal to $\frac{n}{m}$, it is, however, not necessary to set off these quantities separately. **The common tangent of the pitch circles must pass through O_{bc},** as shown by the dotted line in Fig. 65. The centrodes

[1] The teeth of *d* must here be counted as *two* elements, for *d* is paired both with *b* and *c*, although for convenience sake only one set of teeth is actually used for both pairings, by placing the wheels *b* and *c* opposite each other.

of b and c are still circles, for they are bodies turning with a constant velocity ratio about fixed axes, and this

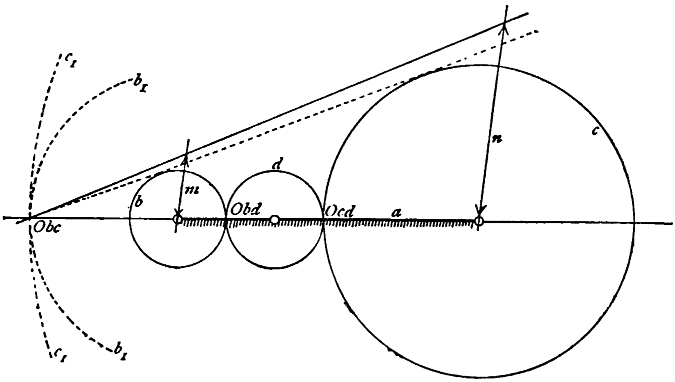


FIG. 65.

remains true of every pair of wheels in any spur gearing whatever, unless the wheels are non-circular. The compound chain of four links might therefore, so far as the motions of b and c are concerned, be replaced by a simple chain of three links with b_1 and c_1 for the two wheels, c_1 being an annular wheel. This arrangement is sometimes used in practice, but there are so many inconveniences about it that the use of an idle wheel is generally found more suitable in cases where two wheels have to receive a constant angular velocity-ratio and to turn in the same sense.[1]

Fig. 66 is an example of the other principal form taken by compound spur-gearing. Here the intermediate wheel

[1] Prof. Reuleaux showed, at the Exhibition of Scientific Apparatus in 1876, a very ingenious arrangement for doing this without either an annular or an idle wheel, by a mechanism which, although resembling a spur-wheel train, was really an altered form of linkwork. This mechanism is illustrated and described in detail in *Berl. Verhandl.* 1875, p. 294, and *Der Constructeur*, p. 537 (fourth Edition).

d has two sets of teeth of ***different*** radii, one pairing with *b* and one with *c*. The point O_{cd} in *c* has therefore no longer the same linear velocity as O_{bd} in *b*, but a velocity greater

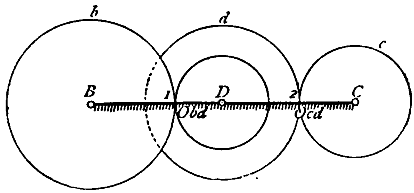


FIG. 66.

or less exactly in proportion to the radii of those points as points in *d*. Instead, therefore, of the ratio $\frac{\angle \text{ r. vel. } c}{\angle \text{ r. vel. } b}$ (the velocity-ratio of the train) being equal to $\frac{B1}{C2}$ as before, it is equal to $\frac{B1}{C2} \times \frac{D2}{D1}$, the points 2 and 1 being points which have the same angular velocity (both being points of the same body *d*), and whose linear velocities are therefore directly proportional to their radii. This ratio can be easily remembered if it is noticed that the numerator is the product of the radii of the driving wheels, and the denominator that of the driven wheels. For the radii, of course, the diameters may be substituted, or the number of teeth, if either happen to be more convenient.

Exactly the same methods as those just used apply to the case of a compound-wheel train with more than three axes. It only requires to be remembered that the first and last wheels *b* and *c* turn in the same or in opposite senses according to whether the number of axes is odd or even respectively.

The general conclusions to be remembered are these: every spur-wheel train transmits a constant angular velocity ratio between its first and last wheels; the value of this ratio is known from the radii or other dimensions of the wheels of the train; the centrodes for the motion of the first and last wheels relatively to each other are therefore circles of known radii, having their centres at the centres of those wheels. For the whole train therefore, whatever wheels it may consist of, there may always be supposed substituted, for kinematic or mechanical purposes, one pair of wheels of known radii and centres, these wheels corresponding to the centrodes of the first and last wheels of the train. If the original gearing contained no annular wheels, and its number of axes was *even*, these centrodes will touch externally,—will correspond, that is, to the pitch circles of spur-wheels. If the number of axes was *odd*, and the original gearing contained no annular wheels, the centrodes will touch internally, that is, the whole train will be equivalent to an annular wheel and a pinion. It is only because it would be frequently inconvenient to use in practice wheels having the diameters of the centrodes, that compound wheel trains are used instead of simple ones. In considering compound trains, however, and especially the compound "epicyclic" trains which will be described in the next section, it may often make apparently complex motions appear much more simple if a pair of wheels such as we have just mentioned be in imagination substituted for the numerous wheels of the actual mechanism.

The virtual centre of the first and last wheels of a train, that is the point of contact of their centrodes, is generally most easily found by calculation from the known velocity ratio of the train, and marked in its proper position on the line joining the centres of the wheels. But a direct

construction for determining this point graphically for any train, when the position of the axes and the diameter of the wheels are given on paper, will be found given in a note at the end of this chapter.

§ 20. EPICYCLIC GEARING.

We have seen that a kinematic chain may be converted into a mechanism in as many ways as it has links (see p. 67), because any link may be made the fixed one. This of course applies as much to the chains we are now considering as to ordinary linkwork, and it is a matter of not at all unfrequent occurrence to find spur-wheel mechanisms in which one of the wheels is the fixed link instead of the frame. Such mechanisms are generally called **epicyclic trains,** because in them one or more wheels revolve about the fixed one, in such a way that points in these wheels describe different cycloidal curves during their motion. When the frame is the fixed link the only kinematic question which commonly occurs is the relative angular velocity of the first and the last wheel, and this we have already considered. When one wheel is fixed, it is generally the relative angular velocity of the last wheel and the revolving arm which we require to know, and this we may therefore now look at. Let Fig. 67 be an epicyclic train in which the wheel b is to be fixed, and let r be the (known) velocity-ratio of the train (page 132), r being positive or negative according to whether c turns in the opposite or the same sense as b when both move, and here therefore positive. Now suppose a to make one complete revolution in either sense, *carrying b with it.* Then c must make one revolution *in the same sense* about its own axis, *simply on account of the*

motion of a and without any action of the wheel-teeth. But the question is to know what motion c would have had had b been stationary. Let b, then, be made stationary, *i.e.* let it receive one turn back in the opposite sense to that in

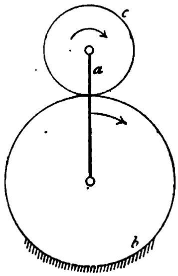


FIG. 67.

which a carried it, so as to bring it into its original position. Then c must necessarily receive r turns in the opposite sense to b, *i.e.* in the same sense as that in which a moved. The whole motion of c, therefore, for one revolution of the arm a, has been—

$$(1 + r) \text{ revolutions.}$$

This is only a re-statement, in another form, of the proposition given at the end of § 16. If c_b and c_a be the angular velocities of c relatively to b and a respectively, and b_a the angular velocity of b relatively to a, then

$$c_b = (c_a + b_a).$$

But $\frac{c_a}{b_a} = r$, hence

$$\begin{aligned} c_b &= (b_a r + b_a) \\ &= b_a(r + 1). \end{aligned}$$

The angular velocity of b relatively to a, (b_a), must be equal to that of a relatively to b, so that we have, just as above,

$$c_b = (r + 1)$$

per unit of velocity (which we have taken as one revolution) of a.

This holds equally true whether the velocity ratio r is obtained by one pair or by any number of pairs of wheels, remembering only that r is a quantity which may be intrinsically positive or negative. The number of revolutions made by the last wheel of an epicyclic train for each revolution of the arm is therefore equal to one *plus* the velocity-ratio of the train if the number of axes in the train

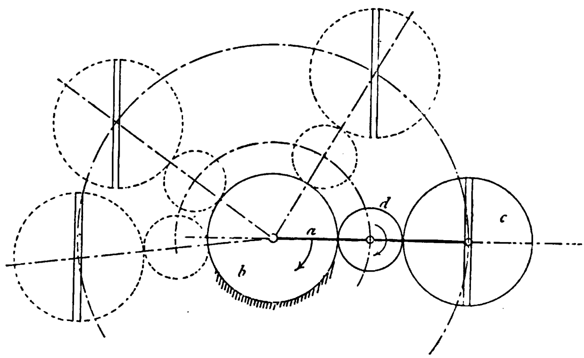


FIG. 68.

be even, and one *minus* the velocity-ratio of the train if the number of axes be odd.[1] In the former case the wheel turns in the *same* sense as the arm; in the latter in the

[1] Assuming that there are either no annular wheels in the train, or an even number. If there be one or any odd number of annular wheels in the train the *plus* and *minus*, and the sense of rotation, are transposed.

opposite sense, unless the ratio r is less than 1. If therefore we employ a train proportioned as Fig. 68, where the velocity-ratio is $= 1$, and the number of axes is odd, we have a case in which c makes $1 + (-1)$ revolutions for each revolution of the arm a, that is, c does not revolve at all. Various positions taken by c are shown in dotted lines in the figure; it will be seen that its motion is translation only, without any rotation.

If the first wheel b of any epicyclic train have its axis fixed, but have itself a motion of rotation about that axis, this must of course be taken into account in considering the motion of the last wheel. If b makes n revolutions for each revolution of the arm, then c makes $1 \pm r \pm nr$ revolutions instead of $1 \pm r$ revolutions, the sign of r being determined as before, and the sign of nr being $+$ if b causes c to rotate in the same sense as the arm, and $-$ if b causes c to rotate in the opposite sense to the arm.

It is not necessary in any way that in such trains as Figs. 66 or 68 the three (or more) wheel centres should lie in one line. If they do *not* do so, of course the line containing the point O_{bc} is always the line joining the centres of b and c, whether that line be the centre line of the frame or not, but the velocity-ratio of the train is in no way changed by altering the position of the centre of c relatively to that of b. The only change often adopted, however, from the forms shown in our figures is that *the centre of the last wheel is often made to coincide with that of the first.* A mechanism in which this is done, of which Fig. 69 shows an example, may be called a **reverted** train. There is no theoretical difference between the trains shown in Figs. 66 and 69, and—as these two have been made intentionally with the same diameters of wheels—the relative angular velocities of c and

a are the same in both cases. The only difference is that if in each *b* be supposed fixed, the rotation of *c* is very much more easily utilised for most purposes when it is placed as in Fig. 69 than when it is placed as in Fig. 66. In each case (with the proportions shown) *c* will make $1 - 2·5 = 1·5$ revolutions for each revolution of the arm, and in the opposite sense to that of the arm's motion.

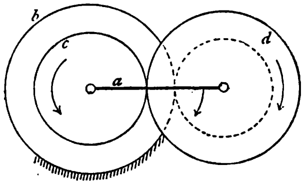


FIG. 69.

The use of a reverted train enables us to transmit with only two axes some velocity-ratios (either very great or very small) which otherwise could not well be transmitted without a much larger number, and even then with considerable inconvenience. Suppose, for instance, we wish to transmit a velocity ratio of 1 : 2,500, *i.e.* we require a driven body to turn only once about its axis while the driving body is turning 2,500 times. We need only make *a* the driving body (it may of course be itself a wheel, and driven by another wheel), and use a reverted train having a velocity-ratio equal to $-\dfrac{2,499}{2,500}$ or $-\dfrac{2,500}{2,499}$ according to the sense of rotation required. The negative sign is ensured by the use of the second axis. The ratio would be obtained by giving *b* 51 teeth, *d* 50 and 49 teeth, and *c* 50 teeth. The velocity-ratio of such a train would be

$$\frac{51 \times 49}{50 \times 50} = \frac{2,499}{2,500} = r$$

and the number of revolutions made by c for each revolution of the arm would be

$$1 - \cdot r = \frac{1}{2,500}.$$

Every ratio could not of course be obtained so easily as 1 : 2,500, but those ratios which cannot be obtained exactly in this way can be obtained approximately, which is in most cases all that would be done if they were obtained by the use of ordinary instead of reverted wheel trains. One of the commonest uses of reverted trains is in the feed motion for the tool carried by a boring bar. Other applications of the same mechanism, commonly with the use of an annular wheel, are to be found in what are called differential pulley blocks, (§ 40), while Reuleaux describes a case in which it has actually been employed as a form of rotary engine.[1]

We thus see that the main problem of the epicyclic train, the finding of the relative angular velocities of the last wheel and the revolving arm, is in all cases a really simple one, involving nothing more difficult than finding the velocity ratio of the gearing as a mere wheel-train and noting the number of axes and whether there are any annular wheels. In a reverted train the central axis must of course be counted twice (as being the axis both of the first and last wheels), so that such a train cannot have less than three axes.

The actual motions occurring in a wheel-train are best followed, if they seem difficult to understand, by substituting for the actual wheels the centrodes of the first and last only. There is one case only in which this substitution of two wheels for the whole train cannot apparently be

[1] *Kinematics of Machinery*, p. 434.

carried out, when, namely, the original train is *reverted*, for of course two wheels having the same axes cannot gear with each other. An examination of what has occurred in this case will show that the three virtual centres of the arm and the two wheels have become coincident in one point, the centrodes also coincide with the same point, and all the construction becomes imaginary. This, however, does not prevent us from dealing with reverted epicyclic trains in the same manner as before, for we have seen that the reversion, —the coincidence of the first and last axes,—was a change adopted purely for constructive reasons, and not one which affected the working of the mechanism in any way, so that there is no reason why we should not change the mechanism back into its normal form, with axes all separated, and in this form work out as before any problems which we have to examine.

Figs. 70 to 72 are examples of this. The mechanism sketched in Fig. 70 is a compound reverted epicyclic train of four links. The link b is fixed, the arm a revolves carrying with it the axes of d and e, the last wheel c has its axis coincident with that of b. The virtual centres O_{ac}, O_{ab} and O_{bc} are in one point, and the centrodes themselves are condensed into the same point, so that constructions involving either are impossible. Fig. 71 shows precisely the same mechanism with its axes separated. The points O_{ab}, O_{ac} and O_{cb} are marked, and dotted circles b' and c' show the form of the centrodes of b and c, the pitch circles of the one pair of wheels which would transmit the same angular velocity ratio as that of the whole train, and which therefore may be supposed substituted for the whole train.

It was unnecessary to have spread out the axes in line as has been here done, nothing is necessary beyond shifting the axes of c and b out of coincidence, as, for instance, has been

done in Fig. 72, where the three virtual centres are again marked, and parts of the pitch circles of the wheels b' and c', together equivalent to the whole train, are shown in dotted lines.

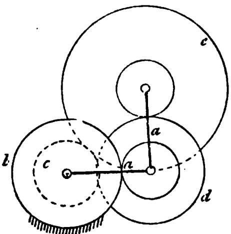


FIG. 70.

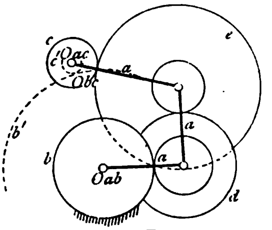


FIG. 72.

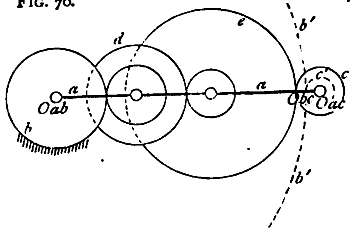


FIG. 71.

In § 40 we shall have further examples of this treatment of reverted epicyclic trains, when dealing with their equilibrium under forces.

§ 21. OTHER MECHANISMS WITH SPUR-WHEELS.

THE mechanisms examined in this chapter have all contained one ordinary link, with turning pairs, as well as the spur-wheels with their higher pairing. We shall now examine a few mechanisms in which the ordinary links play a more conspicuous part than in the former ones, but in which the presence of toothed gearing is still a characteristic.

Fig. 73 shows the well-known "sun-and-planet" mechanism of Watt. The special feature of this chain is that the two equal[1] wheels with the link *a* do not really form part of a common spur-wheel train such as was dealt with in § 17. The outer wheel forms a part of the link *b*, the connecting

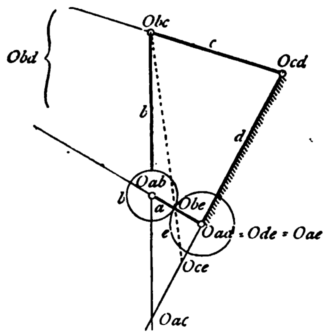


FIG. 73.

rod, and is therefore *not* free to revolve about the axis of the crank-pin. It has, on the contrary, only a swinging motion, not a rotation, in the plane, a motion which approaches the more nearly to a simple translation the longer the rod be made, and which may be treated as a simple translation so far as its effect on *e* is concerned.

To find the motion of *e*, imagine in the first place the wheel *b* to be attached to *a* and to make one revolution with it. The wheel *e* would be in gear with it all the time, and would therefore be carried round one revolution in the same sense. To bring the wheel *b* now back into its original condition, the condition in which it would be if it had been attached to the arm *b*, it must receive one turn *backwards* about its own axis. For in turning it round with *a* it has

[1] By an error in engraving, Fig. 73 shows two *unequal* wheels.

received one turn forwards, and had it been fixed to *b* it would have had no rotation whatever. This backward rotation of *b* will give *e* another revolution *in the same sense as before*. **The wheel *e* therefore makes two revolutions for each revolution of the crank, and in the same sense.**

This double revolution of the shaft comes out in a curious way on examining the virtual centres of the mechanism, some of which are marked in the figure. It will be seen that the virtual centres of *b*, *c*, *d*, and *e* are the same as if these four links together formed a lever-crank mechanism just like the original one, but with a crank of only *half the length*, viz. $O_{ae} O_{be}$, the radius of the spur-wheel.

Fig. 74 shows a mechanism very different in appearance, but very similar in reality, to the one just examined.

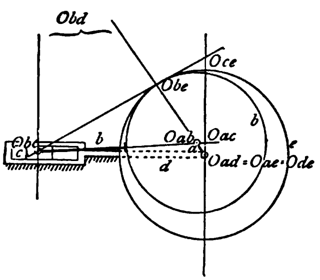


FIG. 74.

It is used for certain capstans. The wheel *e* is here annular, but is, as in the last case, free to turn about the axis of *a*. The short crank *a* in actual construction takes the form of an eccentric. The wheel *b* forms, as before, part of the connecting rod *b*, which here is paired with a block *c* instead of a lever. The mechanism is thus based on the slider-crank

instead of on the lever-crank. In general the wheel *b* is made as nearly as possible equal to *e*; as shown in the figure *b* would have 20 teeth and *e* 24. For every revolution of the crank, therefore, by the reasoning used above (p. 144), the wheel *e* would make $1 - \frac{20}{24} = \frac{1}{6}$ of a revolution, the negative sign being used because the wheel *e* is annular. When employed in a capstan the link *c* and the rod part of *b* are very light, being only required to prevent the rotation of *b*. The link *a* is the driving link, the capstan head being attached to the top of its shaft. The annular wheel *e* is formed upon the bottom of the drum, which therefore rotates, as driven link, freely about the shaft. The drum rotates only one-sixth as fast as the capstan, with, as we shall see presently, a corresponding "mechanical advantage." The only virtual centres which can present any difficulty are marked in the figure.

Figures 75 and 76 illustrate a case in which a mechanism, impracticable if constructed in simple links,

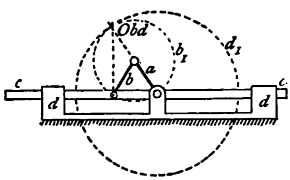


FIG. 75.

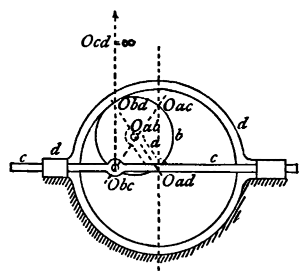


FIG. 76.

may be made constructively convenient by utilising the centrodes of two of them. Fig. 75 shows the original mechanism, a slider-crank in which the connecting-rod *b* is

made equal in length to the crank *a*. This mechanism has this characteristic, that if only it be carried fairly on through a revolution of *a*, the slide *c* will have a stroke *twice* as great as that corresponding to the crank. But unless special means be used to constrain it, as soon as *c* gets to the centre of its stroke, the two ends of *b* coincide with those of *a*, and *b* and *a* together will simply revolve about coincident axes, the link *c* standing still; the whole mechanism will have reduced itself to a pair of elements. A position where such a want of constrainment occurs is called a **change-point.** It is easy to see that the centrodes of *b* and *d* are the two dotted circles b_1 and d_1, the former half the diameter of the latter. If therefore we can only compel these curves to continue rolling on one another, we can constrain the motion we require. Reuleaux [1] has shown a mode of doing this just at the change-points, leaving the mechanism as it is in other positions. A clumsier but more usual method of doing the same thing is shown in Fig. 76, a form of mechanism not uncommon as a "parallel motion" in printing machines and other cases. Here the centrodes are utilised directly as the pitch circles of two wheels, one an annular wheel forming part of *d*, the fixed link, the other a spur-wheel attached to *b*. The radius of this latter is equal to the length of *a* or *b*. Except when the crank *a* is in its mid-position (at right angles to *c*), these wheels do not in any way whatever affect the motion, and might as well be absent. In the mid-position of *a*, however, the gearing of the two wheels determines the point about which *b* rotates to be O_{bd}, the position of which will then be the upper or lower point of the larger pitch-circle, whereas if the mechanism changed into a turning pair (in the way above described) *b* would have to turn about O_{cb}, which would coincide with

[1] *Kinematics of Machinery*, p. 303.

O_{ad}. Reuleaux's method consists essentially in lightening the mechanism by omitting all unnecessary parts of the two wheels, and actually constructing only those teeth which come into gear at the points where constrainment is required.

In the mechanism just described the two wheels formed parts of two already existing links: they did not alter the mechanism at all, but only acted by constraining it in certain positions. In Fig. 77 we return to a mechanism

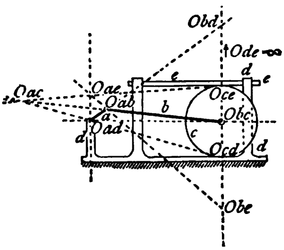


FIG. 77.

in which the spur-wheel plays an independent part, this time in connection with a sliding pair. The crank *a* and connecting rod *b* move exactly as in a slider-crank chain, the latter link being paired with a spur-wheel *c*, which may be of any size. The wheel *c* is compelled to rotate, backwards and forwards, by being geared with a rack upon *d*. On its upper periphery *c* is geared with another rack *e*, which is itself connected with *d* by a sliding pair. The point about which *c* rotates, relatively to the fixed link *d*, must be always at O_{cd}, the lowest point of its circumference. The velocities of the points O_{ce} and O_{bc}, both points of *c*, vary as their distances from O_{cd}, so that the former is always double that of the latter. But O_{ce} is also a point of the slide *e*, all points

of which have the same velocity, as it has a motion of translation only. Hence in this mechanism the velocity of the slide *e* is in every position double what it would be in an ordinary slider-crank chain having the same crank and connecting rod. The length of stroke of *e* is, of course, also double what it would be in that case.

The same mechanism could also be arranged as in Fig. 78, so as to make $\frac{OE}{OB}$ any required ratio. The link *c* in these cases is exactly in the position of any rolling

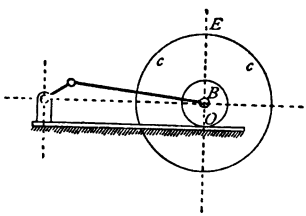


FIG. 78.

cylinder, turning about the point O on which it rolls. An ordinary railway wheel is in exactly this condition. Relatively to the carriage frame it turns about the axis of the shaft; *relatively to the earth*, however, *it turns about the line along which it touches the rail.* Any point on the periphery of the wheel, therefore, which is level with the shaft, is moving at an angle of 45° upwards or downwards. The topmost point of the periphery is for the instant moving in the same direction as the train, but with twice its velocity. The lowest point on the *flange* of the wheel tyre, which is *below* the level of the top of the rail, is moving in the *opposite* direction to the train, and at a velocity proportionate to its distance from the surface of the rail. All this

follows as a mere matter of course from the fact that the wheel is turning about its line of contact with the rail; but it is not easy to realise, on watching the passing of an express train, that some points of each wheel are moving forward twice as fast as the train, while others are actually moving in the opposite direction relatively to the observer.

We have supposed, in this and the preceding section, that spur-wheels were used to communicate a *constant* velocity ratio, and have seen that in this case the spur-wheels were circular. But mechanisms exist, although they are not common, in which spur-wheels are used to communicate some *varying* velocity ratio between two bodies revolving about fixed axes. Suppose we had a mechanism consisting simply of two such bodies, *b* and *c*, with the frame *a* connecting their centres, as in § 17 (Fig. 58). Considering, then, the frame *a* as fixed, we can show, by identically the same proof as before, that the point of contact of the centrodes of *b* and *c* must always lie on the line of centres. But as the velocity ratio is *not* now constant, it will not occupy always the same position on that line, but will be continually changing its distance from the two fixed centres. The centrodes of *b* and *c* must therefore be *non-circular* curves, although they must still roll on one another, revolve each about its proper fixed centre in *a*, and touch on the line of centres just as before. The required motion could, just as before, be transmitted by (non-circular) friction wheels, but for the same reasons as formerly toothed wheels are used, their pitch curves being not now circles but the non-circular centrodes corresponding to the required motion. On these curves teeth can be constructed, by the method of § 18 (or otherwise), just as before, and teeth so constructed will constrain a relative motion corresponding exactly to the rolling of the centrodes—that is, to the required varying

angular velocities of the bodies. A mechanism such as that of Fig. 119 sometimes appears in the form of elliptic toothed wheels, the centrodes of *a* and *c* relatively to each other being ellipses (see Figs. in § 42). The link *d* is in this case generally fixed, and the link *b* always omitted. In its link-work form this is one of the mechanisms which has a change-point, this point occurring when all four links are in a straight line, from which position motion may continue with *b* and *d* either parallel or crossed. Reuleaux has proposed to constrain the mechanism at this point by using what are virtually very small segments (single teeth) of the elliptic wheels, sufficient to fix the position of the virtual centre at the only point where it could change, in the manner already described in this section in connection with Fig. 76.

§ 22. CAM TRAINS.

A non-circular cylinder or disc, its periphery formed of circular arcs or any other curves, used to give motion to some link of a chain with which it is connected by higher pairing, is called a **cam**. The drawback of the line contact of higher pairs (p. 57), and their consequent rapid wear, is felt here much more than in toothed wheels, and for this reason, and others connected with it, mechanisms containing cams, which we may call **cam trains,** are not very common in ordinary engineering work.

Within certain limits the use of cams allows of the certain transmission of velocities varying very widely and in an easily determined fashion, from a uniformly revolving shaft, and they are thus often enough very convenient.

It will suffice if we here deal with two or three examples of cam trains, of which one, a train sometimes used in shearing or punching machines, is sketched in Fig. 79. This

mechanism contains three links only, a fixed link or frame *c*, a shearing lever *b*, and a cam *a* turning about a fixed point on the frame, and driving the lever by pressure on its under side. In order that the mechanism may be constrained, the lever must be kept pressed against the cam either by its own weight or by some mechanical arrangement. The points O_{bc} and O_{ac} can be marked at once. The third virtual centre, O_{ab}, must be in the same line with them. Its position is at once determined by drawing

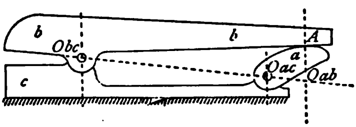


FIG. 79.

a line through the point of contact, *A*, of *b* and *a*, normal to their surfaces. For the point *A*, as a point of *a*, can move relatively to *b* only in the direction of the tangent to the contact surfaces. The point about which *A* is turning must therefore be upon a line through *A* at right angles to those surfaces.

Fig. 80 again shows a cam train of three links, but one of the three pairs is here a sliding pair, a more usual combination than the last. The three virtual centres lie on a line (horizontal in the Figure) at right angles to the direction of motion of *b*. O_{ab} is found as in the last case. O_{bc} is infinitely distant.

If a part of the cam's periphery, as in the case before us, be a circular arc struck from the centre about which the cam rotates, the link *b* will remain stationary so long as this part of the cam is sliding along it. During this time the

contact between *a* and *b* will be in the line of the axis of *c*, and O_{ab} will have become coincident with O_{ac}.

Fig. 81 shows a somewhat more complex case where there are four links, arranged as in an ordinary slider-crank

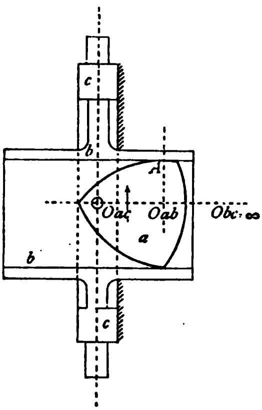


FIG. 80.

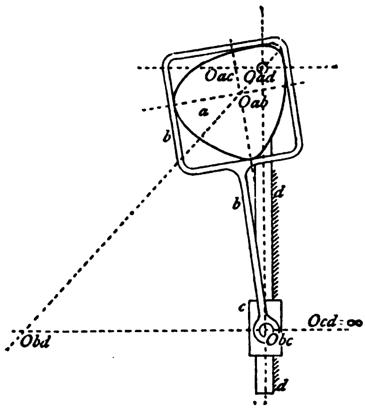


FIG. 81.

mechanism, but with a cam taking the place of the crank-pin.[1] The virtual centres are marked in the figure. The whole mechanism is equivalent to a slider-crank with a crank of varying length, this length being in each position the distance O_{ad} O_{ab}.

Reuleaux has examined at great length and with much ingenuity the form to be given to cams such as those of Fig. 81, so that they may work as elements of higher pairs in complete constrainment, and not, as in Fig. 80, con-

[1] *Not* the place of the crank itself, for the link *a* has two elements, one the cam, the other the shaft paired with the frame *d*. For the real shape of the cam in this case (which is very imperfectly shown in the Figure), see *Kinematics of Machinery*, Figs. 101, 114, etc.

strained only by the arrangement of the rest of the mechanism. This matter we shall not, therefore, examine here. But we shall take one example of cam design from a quite different, but equally important, point of view. Let the problem be the proportioning of a cam such as that of Fig. 79, and the data that the cam is to revolve about a centre O, and drive a straight bar in such fashion that the positions OA, OB, OC, OD of any radius of the cam correspond to positions a, b, c, d of the driven bar. Further, let a be the lowest and c the highest position of the bar, and assume that it is required to remain stationary in its lowest position while the cam continues its rotation from OD back to OA. Let us start with position A. Here the point 1 must be the point of contact and $O1$ the radius of the cam. For, if the point of contact were anywhere else along a, further revolution of the cam in one direction would *lower* the bar, and this we do not wish to do. Further, as the bar is to remain at a during a certain period, the cam's form must be that of a circular arc with radius $O1$ during the corresponding angle, which we can draw by making the angle $1O4$ equal to the angle DOA. When the bar is at b, the radius $O1$ is at $O1'$, the angle $1O1'$ being $= AOB$. The position of the point 2, where the cam is to touch the bar when the position b is reached, is not absolutely, that is uniquely, fixed, but it must fulfil certain conditions; and if the relative positions A and a, B and b, &c., are taken quite arbitrarily in the first instance it by no means follows that these conditions can be fulfilled. In other words, it is not possible by means of a cam working against a straight bar to give the latter *any* series of arbitrarily chosen positions for *any* series of arbitrarily chosen angular positions of the cam. The point 2, if any such point can exist, can be found under the following conditions:

(1) the radius $O2$ must not be *greater* than $O3'$, for, if it were, c would not be the highest position of the bar; (2) it must not be *less* than the distance from O to the line b; and (3) when the cam is in the position shown—*i.e.* with its lowest point in contact—the point 2 must lie below the line a, for otherwise the bar could not touch the cam at 1.

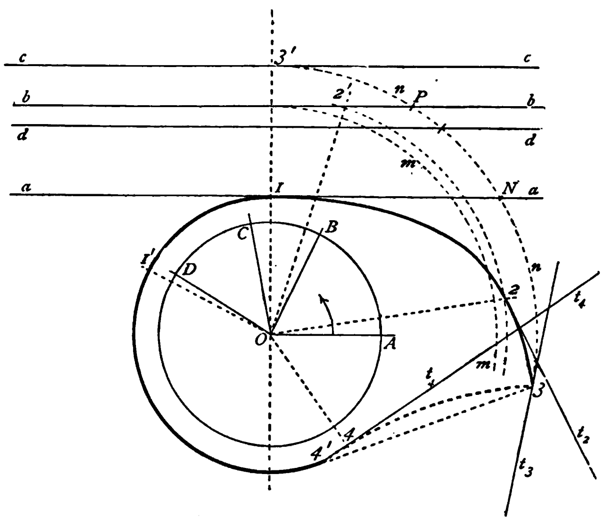


FIG. 82.

Its position when the cam is in the position A must therefore be somewhere between the circles m and n, and below the line a. Further, it is conditioned by the fact that, from whatever position it starts, it must come to b after the cam has turned through the angle BOA. Should the angle PON be greater than the angle BOA it is obvious that this cannot be done, even taking the extreme radius ON for the

point 2, and making the cam straight from 1 to N, in which case, directly contact ceased at 1, N would become the driving-point, and would so continue until it reached 3′. Under the conditions shown in the figure it is unnecessary to go to such an extreme. We may take 2 within the radius n, but in order to find within what limits we can choose its position we must first mark the point 3, which is uniquely determined. For this point must lie on the circle n, and when the cam is in its A position it must be back from OA by an angle $AO3 = CO3'$, the point 3′ being itself fixed as the point 1 was on p. 154. The determination of the point 3 gives us the condition that the tangent at 2 must pass *outside* 3. For otherwise 3 would come in contact with b before 2 could do so, and the intended motion of the cam could not be obtained. This tangent t_2 can always be drawn as a line making the same angle with $O2$ that the line b makes with $O2'$. As a last condition, the angle $2O2'$, passed through by the radius $O2$ between the A and the B positions of the cam, must be equal to the given angle AOB. In the figure the given relative positions of the cam and the bar have been so chosen that all these conditions can be fulfilled and the point 2 found. It is then only necessary to join the points 1, 2, and 3 by any fair curve convex outwards and having the required tangents a, t_2, and t_3 at the three points. In the extreme case 2, 3 may be a straight line, a part of the tangent at 2, which will cause the point 3 to become the driving-point directly contact at 2 ceases.

The point 4, corresponding to d, has to be found by a similar construction, and its possible positions are limited by similar conditions, but it can easily be seen that these conditions can *not* be fulfilled in our example. For the tangent t_4 to the cam at 4 (when position a is again reached) falls within the point 3. This will make the outline or profile of

the cam *concave* outwards between 3 and 4 (as dotted). The point 3 will therefore still remain the driving-point after the position C is passed, and will not only fix the D position of the bar in some other place than that chosen, but will prevent the bar reaching its lowest position until the point 4′ (the point where a line drawn from 3 touches the circle) becomes the driving point.

NOTE TO CHAPTER VI.

VIRTUAL CENTRES IN WHEEL-GEARING.

In § 19, it was pointed out that the virtual centres of wheel-trains could be determined graphically and without calculation, although in most cases this was not the most convenient method. The constructions connected with this matter have, however, sufficient geometrical interest to justify their being given here. Let Fig. 83 represent an

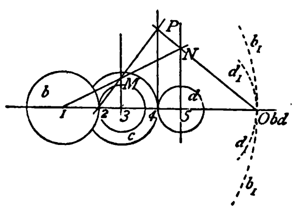


FIG. 83.

ordinary spur-wheel chain of four links; the problem is to find the point O_{bd}, the virtual centre of the last wheel d relatively to the first wheel b. We know, to start with, the diameters of all the wheels (given on the paper), and we know also that the centrodes of b and d must be circles, and that O_{bd}, their point of contact, must lie on the line joining their centres. Suppose b fixed, the frame a turns about the point 1 ($= O_{ab}$).

Let $3M$ represent *on any scale* the velocity of the point $3(=O_{ac})$ of the link a. Then by drawing the line $1M$ we obtain $5N$ as the velocity of the point 5 (which is also the point O_{ad}) in a. The point 2 is the virtual centre O_{bc} of the link c relatively to b. All points in c are therefore moving (relatively to b) about this point, and their velocities vary as their distances from it. We know the velocity of one point in c, namely $3M$, the velocity of 3, for this is the common point O_{ac} of c and a. By drawing $2M$, therefore, we can find at once $4P$ as the velocity of the point 4 in c relatively to b. But the point 4 is O^{cd}, the common point of c and d, therefore $4P$ must also be the velocity of the point 4 in d relatively to b. We therefore know the velocities of two points in d, namely 4 and 5, relatively to b. But the velocity of all points in d relatively to b vary as their distances from the as yet unknown point O_{bd}. To find this point, we have then only to join PN, and mark the point where this line cuts the axis as the required point. The centrodes of b and d are therefore the dotted circles, b_1 and d_1. Writing o for O_{bd}, the velocity ratio of the train is $\frac{10}{05}$, which is intrinsically negative, as 05 is measured in the opposite sense to 10. The given compound train is equivalent to a simple train with two wheels, one of them annular, the

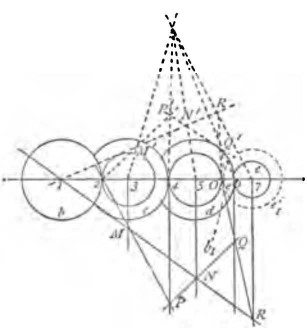


FIG. 84.

pitch circles of these wheels touching in o, and their centres being those of the first and last wheels in the original train.

About this construction we may now notice that the velocity of the point 3 of a, $3M$, was set off *on any scale.* $3M$ was therefore a line of

any length whatever, and we did not actually require to know the velocity of *a*. We have simply to draw $3M$, $4P$, and $5N$ parallels, $1N$ and $2P$ *any* lines from 1 and 2 passing through a common point M, and then join PN to find o.

The construction can be extended with the greatest ease to any number of axes. In Figure 84 one more wheel, *e*, is added, touching *d* in the point 6, and having its centre at 7. Then we know the velocity of two points in *e* relatively to *b*, for the point 7 is also a point in *a* ($= O_{ae}$) and must have the velocity $7R$, while the point 6 is also a point in *d* ($= O_{de}$) and must have the velocity $6Q$. The virtual centre of *e* relatively to *b*, (O_{be}), is found at once, by joining RQ with the centre line of the mechanism. Calling this point o′, the velocity ratio of this train is $\frac{1o'}{o'7}$, and is positive, so that both the representative wheels (shown in dotted lines b_1 and e_1) are spur-wheels, externally toothed.

The relation between the six points 1, 2, 3, 4, 5, o of Fig. 83 leads us to a still simpler construction. For by hypothesis

$$\frac{1 \cdot 2}{2 \cdot 3} \times \frac{3 \cdot 4}{4 \cdot 5} = \text{velocity ratio of train} = -\frac{1 \cdot 0}{0 \cdot 5}$$

$$\text{therefore } \frac{1 \cdot 2}{2 \cdot 3} \times \frac{3 \cdot 4}{4 \cdot 5} \times \frac{5 \cdot 0}{0 \cdot 1} = -1$$

Similarly, in the more extended train of Fig. 84 (substituting the virtual for the actual centres of the first part of the mechanism) we have

$$\frac{1 \cdot 0}{0 \cdot 5} \times \frac{5 \cdot 6}{6 \cdot 7} \times \frac{7 \cdot 0'}{0' \cdot 1} = -1.$$

Six points related in this way are said to be in an *involution*, and from the properties of the involution we may draw certain conclusions affecting the wheel train. In the first place it follows that

$$\frac{4 \cdot 2}{2 \cdot 3} \times \frac{3 \cdot 1}{1 \cdot 5} \times \frac{5 \cdot 0}{0 \cdot 4} = -1,$$

$$\frac{1 \cdot 2}{2 \cdot 0} \times \frac{0 \cdot 4}{4 \cdot 5} \times \frac{5 \cdot 3}{3 \cdot 1} = -1, \text{ and}$$

$$\frac{1 \cdot 5}{5 \cdot 3} \times \frac{3 \cdot 4}{4 \cdot 2} \times \frac{2 \cdot 0}{0 \cdot 1} = -1.$$

In the first two equations given above (p. 159) it will be noticed that the first figure in each numerator and the second in each denominator are the wheel centres, and the others the points of contact. The three equations just given may be taken to represent wheel-trains in just the same way. Thus from the first of them we see that we could have a wheel train with centres at 4, 3, and 5, and points of contact at 2 and 1, and that such a train would correspond exactly to one of two wheels with centres at 4 and 5 and a point of contact at 0, and so on.

Secondly, a matter which is more important for us, all the properties of an involution are *projective*, so that the lines $3M$, $4P$, $5N$, &c., in the figures are not essentially parallel, it is only necessary that they should meet at some one point. They are parallel if that point be taken at an infinite distance, but there is no reason why it should not be taken at any convenient position on the paper. If this be done the construction takes the very general form shown in Fig. 84, which is lettered as the former figures, and needs no further explanation.

CHAPTER VII.

DYNAMICS OF MECHANISM.

§ 23. LINEAR AND ANGULAR VELOCITY.

We have now completed our examination of the nature of mechanisms, as well as of a series of the principal kinematic problems connected with them. So far we have been able to work by methods which are in reality purely geometrical, and have not found it necessary to introduce questions of force or mass at all except to show that they might—for that part of our work—be put upon one side. We now leave the Kinematics for the Dynamics of Mechanism, and come to problems which involve directly questions connected with forces and the balancing of forces. Before we are in a position to deal with these problems it will be necessary to give some further attention to the meanings and relations of the ideas which are involved in them, velocity, acceleration, force, mass, and so on.

We have hitherto looked at motion merely as change of position, and in reference to velocity we have only noticed the relative (and not the actual) velocities of different points in the same body or mechanism at the same time. We have now to deal with problems which involve the determination of actual velocities and the velocities of the same

point at different instants, problems which cannot be understood or solved without continual reference to the forces causing motion or change of motion.

We have distinguished (§ 15) between linear velocity and angular velocity. The former has to do with the distance, measured in ordinary length units, as feet, traversed by a point in a given time, while the latter is measured by the angle swept through by the radius of the point in a given time. Although something has already (in § 15) been said about the relation between linear and angular velocities, it is necessary to look at it here somewhat more in detail.

We have already seen (§ 7) that every body which has plane motion must be rotating about a point[1] in the plane. If this point be infinitely distant (like the virtual centre of a sliding pair), all the points in the body are moving at every instant in parallel straight lines and with the same velocity. In this motion of translation (which is thus merely a special case of motion of rotation) the velocity of the body is fully known if that of any one of its points be known. *The rate of motion of any point in its given direction of motion is the linear velocity of the point*, and in this case, to which we shall in the first instance confine ourselves, the linear velocity of any point is also the linear velocity of the whole body.

When we say that a body has a (linear) velocity 5, we mean that it moves at a rate which, if continued unchanged for a unit of time, would carry it through five units of distance in the given direction. The units of time and distance are commonly seconds and feet respectively, so that in the case supposed we should mean that the rate of motion

[1] The body, of course, rotates about an *axis*, but we have seen that the point which is the intersection of this axis with any plane parallel to the plane of motion, may be taken instead of the axis if we take instead of the body its section by or projection on the plane. See § 9.

of the point was such as would carry it through five feet in a second if it continued unchanged for that period.

But from the mere statement that a body has a linear velocity 5 *at a given instant*, we cannot infer that it possesses that velocity during any length of time, that, for instance, it will actually move five feet in a second, or still less that it will move 300 feet in a minute. We know only that at one particular instant it was moving at a rate which, if continued without change for a second, would have carried it through five feet in that time. This is not in any way inconsistent with its actual movement in the second being three feet or twenty feet instead of five, for its velocity, or rate of motion, may change altogether before the second is finished.

It has to be particularly remembered also that such a numerical value as that just given refers only to the *magnitude* of the velocity, and does not give any information about its *direction*, which, as we shall see, is in many cases equally important.

The velocity of a body is thus what may be called an "instantaneous" quantity. *At a certain instant* the body is moving at such and such a rate. The fact that this rate was quite different the instant before, and will be again quite different the instant after, does not in the least affect the matter. When, therefore, we represent the (instantaneous) velocity of a body by a line A A_1, we do not mean that the body actually moves from A to A_1 in a second, but only that when it is at A it is moving at a rate and in a direction which would bring it to A_1 in a second if only rate and direction continued unaltered for so long.

It happens, however, that we have often to concern ourselves with the *mean* velocity which a body has during a certain interval of time, that is to say, the mean of the

velocities which it has at each successive instant throughout that time, instead of its velocity at one instant only. But a velocity over any interval of time may be *uniform* or *varying*. In the former case the velocity is the same at every instant, and the mean velocity during the whole time is one and the same with the instantaneous velocity at any instant whatever during the time. If a body have a uniform velocity v_0 and pass through a distance s in t seconds the relation between the three quantities is simply

$$s = v_0 t.$$

This relation is equally true if v_0 be the *mean* velocity of a body which has had varying velocity during the time t; when, that is, its velocities at different instants during the time have been different. But in that case the *actual* velocity might be v_0 at perhaps only one instant during the whole time, differing from it more or less at all other instants.

In the case of a varying velocity the rate of variation may itself (as we shall see in § 24) be uniform or varying. In the former case the mean velocity is the arithmetical mean between v_1, the initial velocity, and v_2, the final velocity,[1] or

$$v_0 = \frac{v_1 + v_2}{2} \qquad s = \frac{(v_1 + v_2)t}{2}$$

In such a case the mean velocity is therefore very easily found, and the actual velocity at any instant scarcely less easily. Fig. 92 (on p. 198) is a velocity diagram for such a case, where $v_1 = 1{\cdot}5$ ft. per second and $v_2 = 5$ ft. per second. v_0 is therefore $3{\cdot}25$ feet per second, which is the actual

[1] If the sense of v_2 is opposite to that of v_1 it must have the minus sign prefixed to it, and $v_0 = \frac{v_1 + (-v_2)}{2} = \frac{v_1 - v_2}{2}$

velocity at the end of half the time interval, or two and a-half seconds from the start. v_0 is of course the mean height of the line whose ordinates represent the velocities.

But in the case of a body moving with some irregularly varying velocity, such as that shown by the diagram (Fig. 90) on p. 194, the mean velocity can only be found approximately by taking the mean of the actual velocities at a sufficient number of different points. The arithmetical mean of the initial and final velocities may, in such a case, differ to any extent from the real mean, and could not be substituted for it.

It is very important in what follows that the distinction which we have just enforced between the velocity of a body at a given instant and the mean of its velocities at a number of successive instants, should be kept in mind, a distinction which applies equally to angular and to linear velocities.

The linear velocity of a body, as a quantity having magnitude, sense, and direction, is a "directed quantity," or vector, which has been our justification for the representation of velocities by lines having just those properties, and which justifies us in applying all the graphic rules of vector addition, &c. to lines representing linear velocities.

We have now to look at the case where a body (plane motion being always presupposed) is *turning about a point at a finite distance*, so that its motion is a simple *rotation*. Here, as we have seen, the linear velocity of every point is proportional to its radius, so that all points not having the same radius have different linear velocities. But although the points of a rotating body have so different linear velocities, yet as long as the form of the body is not itself undergoing change, all points in it move through the same angle in the same time. Otherwise, as we said in § 7, different parts of it must have had different motions, and this is

impossible as long as the body remains rigid. The fact that all the points of a rotating body move through the same angle in the same time is expressed by saying that every point in it has the same angular velocity.

Just as either a foot, a yard, or a mile might be used as a unit for linear motion, so several different units might be used for angular motion—a revolution,[1] for instance, or a degree. There are practical conveniences, however, in taking for the unit of angular motion the angle subtended by an arc whose length is equal to its radius, which is $\left(\frac{360}{2\pi}\right)^\circ$ or about 57·3°. As the circumference of a circle of radius r is $2\pi r$, the number of units equal to r in one complete revolution is $\frac{2\pi r}{r} = 2\pi$, which is numerically equal to the distance moved through in one revolution by a point at unit radius. Further, if the body make n revolutions per second, it moves through $2\pi n$ angular units per second, which is again numerically equal to the number of feet traversed per second by a point at unit radius.

The number $2\pi n$ is called the angular velocity of the body, an expression which may be understood to mean either the rate at which the whole body is turning about its axis, expressed in angular units per second, or the rate at which any point in it having a radius equal to unity is moving, expressed in feet per second. The velocity in feet per second of any point in the body whose radius is r feet is obtained by multiplying the angular velocity of the body by the radius of the point, and is therefore equal to $2\pi n r$.

Where a body has a motion of translation only, it is often

[1] In cases where the revolution is used as the unit of angular motion the time unit is most commonly a minute instead of a second.

sufficient for us to take the velocity of *any* point as representing that of the whole body, just as if the whole body were concentrated at that point. But where the body is in rotation about a point at a finite distance, and in all problems which involve the action of forces on the body, and consequently involve consideration of its mass, we may suppose the whole mass to be concentrated only at any point among those which have one particular radius. This radius we may call the **radius of inertia** of the body,[1] and any point having this radius may be called a **centre of inertia** of the body. We cannot take its existence for granted without proof, and the proof will be found in § 32. **Its position is such that if the whole mass of the body were concentrated there in one small particle,[2] the action of any forces on that particle would be the same as their action on the whole actual body.**

The radius of inertia is not equal to the virtual radius of the centre of gravity, and indeed becomes widely different from it when the virtual radius is small relatively to the dimensions of the body.

If, therefore, R be the radius of inertia, in feet, of a body revolving about a given point (whether a virtual or permanent centre) with an angular velocity $2 \pi n$, the body may be represented by a particle of equal mass to itself having a linear velocity $2 \pi n R$ in a given direction, or normal to a given radius.

The linear velocity of any point in a rotating body is

[1] The terms "radius of gyration" or "radius of oscillation," which are sometimes used for it, are, unfortunately, very inconvenient.

[2] The "particle" is supposed to be indefinitely small, so that its size may be entirely negligible. In the case of a body rotating about its own centre of gravity, a thin cylinder or ring takes the place of this ideal particle.

thus a **moment**, and is numerically equal to the product of the angular velocity possessed in common by all the points of the body and the radius of the particular point in question.[1] The linear velocity of a point in a rotating body may therefore be represented by an *area*. It will be numerically equal to twice the area of the triangle whose base is the radius of the point and whose height is the angular velocity of the body.

Thus in Fig. 85 let a body be turning about O with an angular velocity v_a. The linear velocities of the points

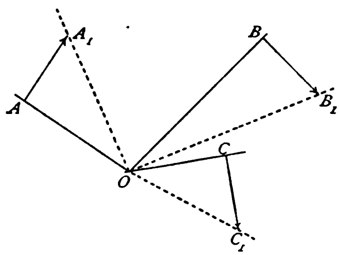


FIG. 85.

A, B, and C are proportional to the areas of the triangles AA^1O, BB^1O, and CC^1O, if AA^1, BB^1, and CC^1 are each made equal to v_a, and set off at right angles to their respective radii. The numerical value of the linear velocity of each point is obtained by doubling the area of the triangle in each case.[2]

In general the different points of a rotating body are

1 It is presupposed that the units of linear and angular velocities are those stated above.

2 If we applied this to the case of translation we should have the radius of every point infinitely great and the motion of the body measured in angular units infinitely small. The linear velocity would, therefore, come out in the form $\infty \times 0$, which cannot be further utilised directly.

moving in different directions, only those lying on the same radius having the same direction, but every point (§ 7) is moving at right angles to its own virtual radius.

§ 24. LINEAR VELOCITY—TANGENTIAL ACCELERATION.

So far as we know, a body which is at rest will remain always at rest, a body which is in motion will move for ever in the same direction with unchanged velocity, unless some extraneous cause alter the condition of rest or of uniform motion. Any such change is called an **acceleration**, and the "cause" producing acceleration is called **force**. It is necessary that the meaning and relations of both these expressions should be examined in some detail, and in the present and next following sections we shall consider the former of them.

A velocity[1] has magnitude, sense, and direction. **Any or all of these may undergo change, and any such change is called an acceleration.** But a change in sense is really only a change in magnitude. If, for instance, a body be moving with a velocity of 10 in a given direction and sense, and its velocity be changed to 5 in the same direction but in the opposite sense, we can say that its velocity has been changed from $+10$ to -5, and therefore the whole change is -15, and can be entirely measured as a change of magnitude. We may therefore say that acceleration must be of one or other (or both) of two kinds, one affecting the magnitude and the other the direction of a

[1] See also §§ 14 and 15.

velocity. The first we shall call **tangential acceleration,** and the second **radial acceleration.** We shall in this section consider only the way in which changes in the magnitudes of velocities, or tangential accelerations, are related to each other and measured.

If the velocity of a body change from five feet per second to eleven, the magnitude of its velocity has been increased by six feet per second, that is, it has received a certain tangential acceleration. It is very important to notice, however, that we do not therefore say that it has received an acceleration of six feet per second, any more than we would say that the original velocity of the body was five feet. A foot is a unit of distance—a foot-per-second is the unit of velocity; and for acceleration our unit must be not a foot-per-second, but a *foot-per-second of velocity added in one second,* or more shortly a **foot-second per second.** A finite increase of velocity must have occupied some finite interval of time, say one second, or three. But there is just as much difference between an increase of velocity which occupies only one second and one which is spread over three, that there is between the traversing of a certain distance in one second or in three. In the latter case the *velocity* is in the one instance three times as great as in the other—in the former case the *acceleration* is in one instance three times as great as in the other.

We have seen in the last section that we may have either to do with the instantaneous velocity which a body actually possesses at a given instant or with the mean of its velocities during a certain succession of instants. We have now exactly the same distinction to make in the case of accelerations. The acceleration which a body is undergoing at a given instant is *the rate at which its velocity is changing at that instant,* measured in foot-seconds of additional velocity

(positive or negative) per second. It does not follow[1] because a body has at a given instant an acceleration of six foot-seconds per second that therefore it will actually in any one second receive this additional velocity. All that we can say about it is that *if* the rate of change of velocity continued unaltered for a whole second the amount of the change would be six feet-per-second.

It frequently happens that our problems are connected not so much with the actual acceleration of a body at any given instant, as with the average value of its accelerations at each instant over some finite time-interval. In such a case we find the total change of velocity which has occurred,—that is the difference between the initial and final velocities,—and divide by the time in seconds to obtain the *mean acceleration during the time.*

The acceleration thus found is not necessarily the actual mean acceleration. It is the acceleration which, if it had acted uniformly for the given time, would have produced the given change of velocity in that time. But if the actual acceleration, as is most often the case, has been varying, it would at most instants differ from the mean acceleration thus found, and it might or might not be a part of our problem to find out at what instants the two values agreed.

The distinction between the actual acceleration at a given instant, and the mean acceleration over a given time, must be kept in mind as clearly as the analogous distinction (p. 163) between instantaneous and mean velocity.

In order, then, that we may measure the real change taking place in the velocity of an accelerated body we must know the rate at which the change is taking place, and our unit for measuring this rate of change, for which "acceleration" is only another name, is one foot-per-second of velocity

[1] See the similar case of velocities on p. 163.

added in one second—or one foot-second per second. The number of units of velocity which would be gained in a unit of time if the change continued uniformly for that time, measures the acceleration of the body, or its rate of change of velocity.

It cannot be too distinctly remembered that we cannot speak of an acceleration of so many feet-per-second. It is unfortunate that we have no short expression to stand for a unit of velocity, so that we are compelled to adopt the somewhat cumbrous phrase already used. If a foot-per-second were called (as Dr. Lodge suggests) a "speed," then the unit of acceleration might be said to be *one speed-per-second.* As it is, a foot-second per second seems the shortest available expression which we can use for it. When, therefore, we say that a body receives an acceleration of 10, we mean that it receives additional velocity at a rate which, if it remained unaltered for one second, would amount to ten feet per second in that time.

When the acceleration of a body is the same at successive instants it is said to be *uniform*, in all other cases it is *un-uniform*, or *varying*. If a body has a uniform acceleration over a certain period of time, its mean acceleration during that period is equal to its acceleration at each and every instant of the period. It is in such a case the same to us whether we have to do with the acceleration at one instant or the mean of the acceleration at many successive instants, for at every instant the acceleration is the same. As this case is so much simpler than that of varying acceleration we shall consider it by itself first, only premising that in a majority of the cases occurring in engineering problems the acceleration is varying, and that the assumption of uniform acceleration in some such cases may lead to serious error, if indeed it does not make the problems altogether meaningless.

If a body moving with a velocity v_1 has that velocity altered to v_2 during a time t, and the acceleration during the whole time is uniform, we shall obtain its value, a, in foot-seconds per second by simply dividing the total gain of velocity by the time which it has occupied, or in symbols thus :—

$$a = \frac{v_2 - v_1}{t}$$

If the body has started from rest, v_1 becomes zero—if it ends at rest, v_2 becomes zero. If the sense of v_2 be opposite to that of v_1, the minus sign must be prefixed to it, and the algebraical difference between v_2 and v_1 becomes their arithmetical sum, as in the example already given on p. 169. But the time t must always be positive, so that in this case a becomes negative.

The equation just written down enables any one of the four quantities which it contains to be determined when the other three are known, thus

$$a = \frac{v_2 - v_1}{t} \quad . \quad . \quad . \quad 1 \qquad v_1 = v_2 - at \quad . \quad . \quad . \quad 3$$

$$v_2 = v_1 + at \quad . \quad . \quad . \quad 2 \qquad t = \frac{v_2 - v_1}{a} \quad . \quad . \quad . \quad 4$$

remembering that (taking the original velocity v_1 as always positive) v_2 may be intrinsically either positive or negative. It must be noticed that either a or t can be found without knowledge of the absolute numerical values of v_2 and v_1, as long as the difference between them is known. It should be noticed also that by putting the equation in the form

$$at = v_2 - v_1$$

we see that at is a *velocity*—numerically equal to $(v_2 - v_1)$ feet per second. An acceleration of so many foot-seconds per second, multiplied by the number of seconds over

which it extends, is equal simply to the whole change of velocity, and is therefore itself a velocity, expressed in feet per second.

If the body start from a state of rest (that is, of no velocity relatively to the standard chosen for the time being, see p. 20), $v_1 = 0$, and the equations become

$$a = \frac{v_2}{t} \quad . \quad . \quad . \quad . \quad 5 \qquad\qquad 0 = v_2 - at \quad . \quad . \quad . \quad 7$$

$$v_2 = at \quad . \quad . \quad . \quad . \quad 6 \qquad\qquad t = \frac{v_2}{a} \quad . \quad . \quad . \quad . \quad 8$$

If a body starting with some finite velocity v_1 come to rest at the end of the change, then $v_2 = 0$, and the equations become

$$a = -\frac{v_1}{t} \quad . \quad . \quad . \quad 9 \qquad\qquad -v_1 = at \quad . \quad . \quad . \quad 11$$

$$0 = v_1 + at. \quad . \quad . \quad 10 \qquad\qquad t = -\frac{v_1}{a} \quad . \quad . \quad 12$$

If, lastly, the body starting with some finite velocity v_1, ends with a finite velocity v_2, having a sense opposite to that of v_1, the equations (as we have already seen), must become

$$a = -\frac{v_2 + v_1}{t} \quad . \quad 13 \qquad\qquad v_1 = v_2 + at. \quad . \quad . \quad 15$$

$$v_2 = v_1 - at. \quad . \quad . \quad 14 \qquad\qquad t = \frac{v_2 + v_1}{a} \quad . \quad . \quad 16$$

The *negative acceleration* in the last two sets of equations means simply that the change of speed has been in a sense opposite to that of the original velocity. A negative acceleration is sometimes called a *retardation*, but the former expression is to be preferred, because (in such cases as those of equations 13 to 16) v_2 may be greater than v_1, although in the opposite sense. If the velocity of a body be changed from 10 to $-$ 20 it can hardly be said to be retarded by the change of $-$ 30, for it is moving twice as

fast at the end of the change as it was at the beginning, although in the opposite direction. We shall therefore call all changes of velocity accelerations, meaning by this that some magnitude has been added to the velocity, whether this magnitude be positive or negative, and whether the final velocities be plus or minus, or greater or less than the initial velocity.

The following examples of problems connected with uniform acceleration illustrate the use of the formulæ of this section :—

A body has a velocity of 50 ft. per second, and receives an acceleration of 5 (foot-seconds per second) for 4 seconds. What velocity will it have at the end of this period?

Here $a = 5$, $t = 4$ and $v_1 = 50$, so that v_2, the velocity required, which is equal to $at + v_1$, is $5 \times 4 + 50$, or 70 feet per second.

Other things remaining the same, what would have been its final velocity if the acceleration had been -7 (foot-seconds per second)?

Here $at + v_1 = -(7 \times 4) + 50 = 50 - 28$, or 22 feet per second. The final velocity is less than the initial velocity as the acceleration was negative.

If a body starts from rest, and acquires a velocity of 21 ft. per second in 6 seconds, what must have been its mean acceleration during that period?

Here $v_1 = 0$, and $a = \frac{v_1}{t} - 21 \div 6$ or $3{\cdot}5$ (foot-seconds per second).

If a body moving at the rate of $17{\cdot}5$ feet per second be brought to rest in 7 seconds what acceleration must it have received?

Here $v_2 = 0$, and $a = -\frac{v_1}{t} = -17{\cdot}5 \div 7 = -2{\cdot}5$ (foot-seconds per second).

How long a time would be required to increase to 120 ft. per second the velocity of a body moving at the rate of 72 ft. per second, if the acceleration were 5 (foot-seconds per second)?

Here $t = \frac{v_2 - v_1}{a} = \frac{120 - 72}{5} = 9{\cdot}6$ seconds.

A body moving at the rate of 270 ft. per second undergoes an acceleration of -18 (foot-seconds per second), how long will it be before it comes to rest?

Here $v_2 = 0$, and $t = -\frac{v_1}{a} = -\frac{270}{-18} = 15$ seconds.

As the equations themselves have not special reference to mechanisms, but are perfectly general, it has not been necessary to choose these examples with any special reference to machines. Their application to engineering problems will be examined further on.

Having seen the principal relations between velocity, time, and acceleration so far as concerns the case when an acceleration is uniform, that is, remains equal at every instant, over a certain time, we have now to look at ***instantaneous acceleration*** (p. 170), or the rate at which the velocity of a body is being changed at a given instant. Let the equal distances OA, AB, and BC in Fig. 86 stand for equal

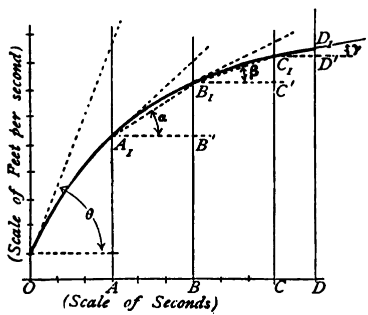


FIG. 86.

intervals of time, each t seconds, and let the ordinates of the curve $A_1B_1C_1$, as AA_1, BB_1, etc. represent the velocities of a body at the times A, B, and C respectively Further let the same distance on the paper stand for a unit of time

and a unit of velocity. Then over any interval such as BC the mean acceleration is approximately

$$a = \frac{v_2 - v_1}{t} = \frac{CC_1 - BB_1}{BC} = \frac{C'C_1}{BC} = \frac{C'C_1}{t}.$$

If we join B_1 and C_1 and call the angle $C_1B_1C' = \beta$, we might therefore put tan β as the mean value of the acceleration over the time interval BC. Similarly tan α is the mean acceleration between A and B. If after C_1 we took any point D_1 *at any time interval whatever* from C_1 we should still get the same result. For between C_1 and D_1 the acceleration would be

$$\frac{DD_1 - CC_1}{CD} = \frac{D'D_1}{CD} = \tan \gamma.$$

We thus get a numerical value for the acceleration in a form in which we can make the time interval as small as we please, or, if we please, zero. For if we take D close to C, then C_1D_1 becomes the line joining two consecutive points of the curve, which we know to be the tangent to the curve, and although in that case we cannot draw the point D', or measure any distance $D'D_1$, yet we know that the line C_1D' must be parallel to the axis. Hence, although we can measure neither CD nor $D'D_1$ we can still get the ratio between them, because we know the angle γ, and therefore know its tangent, which therefore measures the acceleration at the point D, or the instantaneous acceleration.

A curve such as $A_1\ B_1\ C_1$ . . . . whose ordinates represent velocities and whose abscissæ represent times, is called a *velocity curve on a time base.* Calling therefore the angle which a tangent to the curve at any point makes with the base the angle between the curve and the base at that point, we may say that **given a velocity curve on a time**

base, the tangent of the angle between the curve and the base at any point is equal to the acceleration at that point.[1]

In Fig. 86 for instance the acceleration *at starting*, that is, at the point O, is equal to tan θ, or in the figure as drawn 2·5 foot-seconds per second. The acceleration at A_1 is 0·9 foot-seconds per second, and at B_1 0·5 foot-seconds per second, while the approximate mean acceleration between A_1 and B_1, or $\frac{B'B_1}{AB}$, is 0·67 foot-seconds per second.

To avoid possibility of misunderstanding it may be as well to point out that the form of such a velocity curve as that of Fig. 86 has nothing whatever to do with the *path* of the body whose velocities it represents. At present we are supposing that the body is moving always along a straight line, its velocity varying only in speed along that line and not changing in direction at all.

§ 25. LINEAR VELOCITY.—TANGENTIAL ACCELERATION—(*continued*).

In the last two sections we have considered velocity and acceleration only in connection with *time*, and our equations have contained only the three quantities v, t, and a. It is necessary before leaving this part of our subject to consider them also in connection with *distance*, and here of course questions of instantaneous velocity and acceleration no longer occur.

When a body moves with a uniform velocity of v_0 feet per

[1] We shall see in § 28 how to deal with this in the case when (as is usual) the velocities and times are drawn to different scales.

second during t seconds it passes through a distance of v_0 times t feet, a result which we can write

$$s = v_0 t$$

where s is the distance or space in feet passed through in t seconds. When a body, however, moves with a varying velocity during a certain time we can only find out the distance traversed if we know the mean velocity during that time, or the mean rate of change of velocity or acceleration. If the acceleration has been *uniform* the mean velocity is simply the arithmetical mean between the initial and final velocities, or $\frac{v_2 + v_1}{2}$. We shall at first consider only the case of uniform acceleration. The distance moved through in t seconds by a body uniformly accelerated from v_1 to v_2 will therefore be

$$s = \frac{v_2 + v_1}{2}t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We have already seen that

$$a = \frac{v_2 - v_1}{t}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and if we multiply together the two left-hand sides of these two equations, and the two right-hand sides, and double the product in each case we get

$$2as = v_2^2 - v_1^2, \text{ from which}$$

$$s = \frac{v_2^2 - v_1^2}{2a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Further, from the last section we know that $v_2 = v_1 + at$, and putting this value for v_2 in equation (1) above, we get

$$s = v_1 t + \frac{a}{2}t^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Similarly by taking the value $v_2 - at$ for v_1, and putting it into equation (1) we get

$$s = v_2 t - \frac{a}{2}t^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The five equations just given contain five quantities, s, v_2, v_1, a and t, and are so arranged that each one contains four of these quantities, so that any quantity can be found if three of the others are known.

If the body has started from rest, so that $v_1 = 0$, the first equation becomes $s = \frac{v_2}{2}t$. Similarly if the body come to rest at the end of the acceleration, so that $v_2 = 0$, the same equation becomes $s = \frac{v_1}{2}t$.

In these cases it is convenient to write v for the change of velocity, so that $s = \frac{v}{2}t$ and $a = \frac{v}{t}$.

Equations (3) and (4) under these circumstances become respectively $s = \frac{v^2}{2a}$ and $s = \frac{a}{2}t^2$.

For a body starting from or ending at rest we have therefore the four equations:—

$$v = at \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$s = \frac{v}{2}t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$s = \frac{a}{2}t^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$v^2 = 2as \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

These are four equations involving altogether four quantities, v, a, t, and s, arranged so that each equation contains only three of the quantities, and so that any one of the

quantities can be found by the help of one or other equation, from any two of the ,others.

We may now examine examples of these equations so far as they do not fall under those already illustrated in the last section.

A body moving 10 feet per second has its velocity increased to 24 feet per second while it traverses 51 yards. What time does the acceleration occupy?

Here the mean velocity $\frac{v_2 + v_1}{2} = 17$ feet per second, and by (1)

$$t = \frac{2s}{v_2 + v_1} = \frac{153}{17} = 9 \text{ seconds.}$$

A body having initially a velocity of 7 feet per second receives an acceleration of 3 foot-seconds per second for 1 minute. What distance will it pass over in that time?

Here by (4) $s = 7 \times \cdot 60 + \frac{3}{2} 3600 = 5820$ feet. Of course this problem might equally well be solved by calculating v_2 and using equation (1). $v = v_1 + at = 7 + 180 = 187$ feet per second, and

$$s \equiv \left(\frac{187 + 7}{2}\right)60 = 5820 \text{ feet, as above.}$$

A body falls in speed from 9 to 2 feet per second while it is traversing a distance of 70 feet. What is its acceleration?

Using equation (3)

$$a = \frac{v_2^2 - v_1^2}{2s} = \frac{4 - 81}{140} = - \frac{77}{140} = - 0{\cdot}55 \text{ foot-seconds per second.}$$

A body having an initial velocity of 2·5 feet per second receives an acceleration of 1 foot-second per second during 12 seconds. What distance will it pass over in that time? By equation (4)

$$s = 2{\cdot}5 \times 12 + \frac{144}{2} = 102 \text{ feet.}$$

With the same initial velocity and acceleration as in the last case, what time would the body take to pass over 102 feet?

The same equation as in the last case may be used to find t (v_1, a and s being given), but as in this case a quadratic would have to be solved it is

more convenient to find v_2 from equation (3) and then t from equations (1) or (2). From (3)

$$v_2 = \sqrt{2as + v_1^2}$$
$$= \sqrt{210 \cdot 2} = 14 \cdot 5 \text{ feet per second}$$
$$\text{and } t = \frac{v_2 - v_1}{a} = \frac{14 \cdot 5 - 2 \cdot 5}{1} = 12 \text{ seconds}$$

which is a check upon the working of the last example.

A train which has had an initial velocity of 40 miles per hour has had its speed reduced to 5 miles an hour in a minute and a half; what distance will it have travelled in that time?

One mile an hour is 88 feet a minute or 1·467 feet per second, so that v_1 and v_2 are respectively 58·7 and 7·3 feet per second, and from equation (1)

$$s = \frac{58 \cdot 7 + 7 \cdot 3}{2} 90$$
$$= 2970 \text{ feet or } 0 \cdot 56 \text{ of a mile.}$$

How long would it take, under similar conditions, for the train to come to rest, and how far would it have run before it did come to rest?

Here we must first find the acceleration, which we can do from equation (2),

$$a = \frac{v_2 - v_1}{t} = \frac{7 \cdot 3 - 58 \cdot 7}{90} = -0 \cdot 57.$$

Then from equation (6) $t = \frac{v}{a}$, by putting $v = v_1 = 58 \cdot 7$, we find $t = \frac{58 \cdot 7}{0 \cdot 57} = 103$ seconds, and the distance travelled before the velocity becomes zero can be found from equation (7),

$$= \frac{v}{2} t = \frac{58 \cdot 7}{2} \times 103 = 3023 \text{ feet.}$$

The result may also be got from equation (9), which gives

$$s = \frac{v^2}{2a} = \frac{58 \cdot 7 \times 58 \cdot 7}{2 \times 0 \cdot 57} = 3022 \text{ feet.}$$

A body starting from rest attains a velocity of 42 feet per second in 5 minutes. What space will it have passed through in the time?

Here $v_1 = 0$ and $v_2 = v = 42$, and by (7)

$$s = \frac{42 \times 300}{2} = 6300 \text{ feet.}$$

The piston of a Cornish engine, which has a stroke of nine feet, attains its maximum velocity of 12 feet per second at one-third of the stroke: what time does it take in passing through this distance, and what must be its acceleration?

Here $v_1 = 0$, $v_2 = v = 12$. From (7)

$$t = \frac{2s}{v} = \frac{6}{12} = 0{\cdot}5 \text{ of a second,}$$

and from (6)

$$a = \frac{v}{t} = \frac{12}{0{\cdot}5} = 24 \text{ foot-seconds per second.}$$

A train attains a velocity of 50 miles an hour in 10 minutes after leaving a station. What distance will it have travelled in this time and what must have been its acceleration?

Here $v_1 = 0$ and $v_2 = v = 50$ miles an hour $= 73{\cdot}3$ feet per second, and from (7)

$$s = \frac{v}{2}t = \frac{73{\cdot}3 \times 600}{2} = 21990 \text{ feet or } 4{\cdot}16 \text{ miles.}$$

The acceleration can be obtained either from (8) or (9). From the former

$$a = \frac{2s}{t^2} = \frac{2 \times 21990}{600 \times 600} = 0{\cdot}122 \text{ foot-seconds per second,}$$

and from the latter

$$a = \frac{v^2}{2s} = \frac{73{\cdot}3 \times 73{\cdot}3}{2 \times 21990} = 0{\cdot}122 \text{ foot-seconds per second as before.}$$

Two bodies A and B start together with an initial velocity of 4 feet per second; when both are 4 feet from the starting point A has a velocity of 1 foot per second and B of -1 foot per second. What has been the acceleration in each case?

From (3) we have the acceleration as $\frac{v_2^2 - v_1^2}{2s}$ which in the case of both A and B is $\frac{1 - 16}{8}$, or $-\frac{15}{8}$, so that we have the apparent paradox

that the same acceleration of $-\frac{15}{8}$ foot-seconds per second corresponds to different final velocities, although the distances travelled by the two bodies are equal. The paradox is only apparent, however, and can easily be explained. At the end of the process B is moving backwards ($v_2' = -1$) while A is still moving forwards ($v_2 = 1$). So that B must have moved forward *more* than 4 feet, come to rest, and begun to move back again. It will be worth while to work this out, and show that the quantities are in reality quite consistent. First of all, find the time which the operation has taken in each case from equation (1) $t = \frac{2s}{v_2 + v_1}$. This gives $\frac{8}{5}$ seconds for A and $\frac{8}{3}$ seconds for B. (If these values for t are put in equation (2), and the value of a worked out, they will be found both to come to $-\frac{15}{8}$ as above.) Now find how long B must have taken to come to rest, *i.e.* to make $v_2 = 0$. Using equation (6), $\frac{v}{a} = t$, we get the time as $\frac{32}{15}$ seconds. Lastly, finding from the same equation how long it would take to give B, now at rest, a velocity of -1 foot per second, (the acceleration still being $-\frac{15}{8}$ foot-seconds per second,) we get it $\frac{8}{15}$ seconds. Adding these two periods together, we find that to change the velocity of B from 4 feet per second to -1 foot per second, with a uniform acceleration of $-\frac{15}{8}$ foot-seconds per second, would require $\frac{32 + 8}{15} = \frac{8}{3}$ seconds exactly as was calculated above.

§ 26. LINEAR VELOCITY—RADIAL ACCELERATION.

At any one instant a body may, for our present purposes, be represented by a particle having the same mass, and placed anywhere so long as it is at a certain distance (equal to its radius of inertia[1]) from the point about which it is

[1] See p. 167 and § 32.

rotating. For simplicity's sake, let us suppose su to be substituted for the body, and let us further su as we shall see, is far from being true) that for a and motions of the body the position of this re particle in respect to it remains unchanged. At any one instant the particle can be moving in one direction only, change of velocity along that direction, whether positive or negative, is what we have called tangential acceleration. But if the path of the particle be a curve and not a straight line, this direction although fixed at any one instant, varies from instant to instant continually. During such variation the magnitude of the velocity of the body in feet-per-second may remain unchanged, so that there may have been no tangential acceleration. But clearly the *direction* of the velocity has been changed, and direction is not less a property of a velocity than magnitude. Hence, we say still that the body has received an acceleration, or change of velocity, but to distinguish the two cases, we call the acceleration of which we are now speaking, for reasons which will be presently explained, a **radial acceleration.**

But one velocity can be changed into another only by (algebraical) addition of a velocity, and as the added velocity in this case does not (by hypothesis) affect the magnitude of the speed in the original direction, its own direction must be at right angles to the first. When a body,[1] therefore, is moving in a continuous curve it must be receiving at every instant radial acceleration, for if at any instant the radial acceleration became zero the curve would become a straight line. The direction of motion of the body is at any instant the tangent to its path at that instant. Along that

[1] Here and in other places in these sections the word "body" must be understood to mean the *particle* of equal mass with the body already mentioned.

tangent it may or may not be receiving tangential acceleration; that is a point with which we have already dealt in the last two sections. Under all circumstances, however, the body must at every instant be receiving acceleration normal to that tangent, that is, in the direction of its own virtual radius, and this is what we have called radial acceleration. Just, then, as the tangential acceleration of a body at any instant is the rate of change of speed in the direction of motion of the body, so its **radial acceleration is the rate of change of speed at right angles to its direction of motion,** in other words **along its virtual radius,** and equally with tangential acceleration, and for exactly the same reasons, is to be measured in foot-seconds per second, if feet and seconds are, as before, our units of distance and time.

The simplest illustration of radial acceleration is that of a ball swung round a centre to which it is attached by a cord of fixed length. The actual velocity of the ball is (or may be) the same at every instant. If it were left to itself it would move away tangentially to its enforced orbit with just that velocity. But this is prevented by the pull of the string, which compels the ball continually to keep at the same distance from the centre, that is, compels it to be continually nearer the centre than it would otherwise be, although its actual distance from the centre remains unaltered. From this virtual drawing of the body towards the centre of motion radial acceleration is often called *centripetal* acceleration.

One or two points must be noted here which are apt to give rise to misunderstanding. In the first place, although the body, in the case just supposed, is accelerated towards the centre, it does not actually move towards the centre. By hypothesis the acceleration is at right angles to the

direction of motion, and therefore in a direction in which there is no motion. Here, therefore, we have acceleration without velocity. If a body A (Fig. 87) is turning uniformly about a fixed point O, its *velocity* is fully represented by a line AA_1, at right angles to its radius, and its *acceleration* by some line AA_2 along its radius. It has no acceleration along AA_1, and no velocity along AA_2, and care must be taken not to add together AA_1 and AA_2 as if they were similar magnitudes. If the body is not revolving uniformly it will

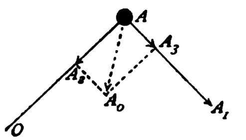


FIG. 87.

have acceleration along AA_1, its direction of motion. If AA_3 were such an acceleration we could add AA_3 and AA_2 together to find the resultant acceleration AA_0, for they would be vectors representing similar magnitudes. But in general it is more convenient to deal with them separately. In any case **the whole acceleration of a body is the sum of its accelerations along and normal to its direction of motion, i.e., the sum of its tangential and radial accelerations** as already defined. This sum can be resolved along and at right angles to any direction whatever, if it is desired to find the acceleration in any given direction. The only direction in which the body will have no acceleration at all will be the direction at right angles to that of its total acceleration.

It is to be remembered that, having dealt with tangential and radial acceleration, we have covered every possible case

which can occur with linear velocity in bodies having plane motion. For all such bodies are either turning about some centre or moving with simple translation. As acceleration is an instantaneous quantity, that is, a rate of change existing at one particular instant, we are obviously indifferent whether the centre towards which radial acceleration occurs is a virtual or a permanent centre. In the latter case, (as with the ball and string,) the acceleration in successive instants is always towards the same centre and normal to a circle. In the former case, the acceleration in successive instants is towards different centres, and at each instant normal to the curve (however irregular) described by the particle which we are supposing to represent the body.[1]

The radial acceleration of a body at any particular instant depends upon the curvature (and therefore upon the radius) of its path and the velocity with which it is moving in that path. We must therefore be able to express the radial acceleration of any body in terms of r and v, and shall proceed to see how this can be done. Let the curve PP_1 (Fig. 88) be a circular arc whose centre is O, and let R

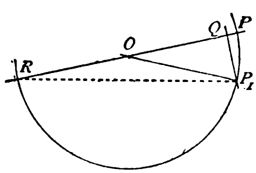


FIG. 88.

be the point on a diameter opposite to P. Let P be a representative particle turning about O, with radius $OP = r$, and therefore moving for the instant in the arc PP_1; further let

[1] As to the representation of a body by a particle, see particularly the last paragraphs of this section, and also the whole of §§ 31 and 32.

the velocity of P (supposed uniform) be v, and the time taken to traverse the distance PP_1 be t, so that $PP_1 = vt$. Let the angle POP_1 be taken so small that arc and chord may be treated as equal. At P the body has no velocity along PO; by the time it has reached P_1 it has moved through the distance PQ in that direction. PQ is therefore the distance moved in the direction PO in a time t by a body starting from rest and undergoing the radial acceleration a which we wish to find. By taking PP_1 sufficiently small we may suppose OP_1 to be (within any desired approximation) parallel to OP, so that the motion PQ is in the direction of OP_1 as well as of OP, that is, is instantaneous radial motion. By equation 8, p. 180,

$$PQ = s = \frac{a}{2}t^2$$

$$\text{so that } a = \frac{2PQ}{t^2}.$$

Further, as PP_1R is the angle in a semicircle, it is a right angle, hence

$$\frac{PQ}{PP_1} = \frac{PP_1}{PR} \text{ so that } PQ = \frac{PP_1^{\,2}}{PR}.$$

$PP_1 = vt$ and $PR = 2OP = 2r$, so that $PQ = \dfrac{v^2t^2}{2r}$ and by substitution

$$a = \frac{2}{t^2} \cdot \frac{v^2t^2}{2r} = \frac{v^2}{r}$$

where v is the velocity of P (in feet per second) along the tangent at the instant when the body occupies the position P, and r is its radius (in feet) in that position. The radial or centripetal acceleration, a, is of course measured in foot-seconds per second, as usual, and its magnitude

varies directly as the square of the linear velocity, and inversely as the radius of the point.

EXAMPLES.—A ball is swung by a cord at 30 inches radius at the rate of 20 revolutions per minute. What centripetal acceleration does the ball undergo?

Here $v = \dfrac{2\pi r \times 20}{60} = 5{\cdot}23$ feet per second, and $a = \dfrac{5{\cdot}23^2}{2{\cdot}5} = 11$ foot-seconds per second nearly. In this question the radius 30″ is the radius of inertia of the ball, and not the radius of its mass-centre (see §§ 30 and 32), although for practical purposes the two centres might often in such a case be treated as identical.

The radius of inertia of a connecting-rod is 5 feet 6 inches, the linear velocity of points upon that radius is 8 feet per second. What radial acceleration has the connecting-rod?

The problem is treated exactly as the last, the fact that O (see *e.g.* Fig. 89) is a virtual instead of a permanent centre making no difference. $v = 8$ feet per second, $r = 5{\cdot}5$ feet, and $a = \dfrac{8^2}{5{\cdot}5} = 11{\cdot}6$ foot-seconds per second. In general in such a problem as this the velocity, v, would have to be found by construction (as in § 14) from a given velocity of crank-pin in revolutions per minute.

We have spoken of the acceleration of a *body*, whose mass was supposed to be concentrated at one point. For many purposes it is sufficient so to treat the matter. But physically the conditions are not so simple. In the first place it is not only as a whole that a body tends to preserve its direction of motion,—each individual particle in it has the same tendency, and of course different particles have very different directions of motion. In such a case as that just supposed, a ball swinging at the end of a string, if the velocity were sufficiently increased the string would break and the ball fly off, spinning round and round at the same time, on account of the greater velocity of its outer than of its inner particles. But if instead of a metal ball swung from a string we had a plaster ball swung by iron links, the

links would not give way under the increased velocity, but the particles of the plaster would separate from each other, and the ball would break up. In the case to be considered later on (see § 48), where such a body as a fly-wheel is revolving about an axis through its own centre of gravity or mass-centre, the wheel, *as a whole*, has no radial-acceleration, but none the less every particle in it has its own radial acceleration, and unfortunately there are not a few cases on record where a too great velocity has so increased this centripetal acceleration that a fly-wheel has burst in pieces with very serious results. The real connection between centripetal acceleration and what is usually (though unfortunately) called the *centrifugal force*, which causes this breaking up, will be examined in § 30.

In the second place, the representation of a body by a particle, dynamically, is only admissible as an instantaneous method. That is, *at any one instant* the whole mass of the body might be concentrated at any point in it having a certain distance from the axis about which the body is turning, this distance being what we have called the radius of inertia of the body. If the body is always turning about the same point, then and then only does this radius remain constant and belong always to the same points in the body. Only under these conditions, therefore, can any one particle *continue* to be representative of the whole body. In the case of a body moving at successive instants about *different* axes, its radius of inertia continually changes, and changes so that no one particle remains at this radius during the change of position of the whole. At each different position of the body, therefore, a *different* particle must be taken to represent it, any one particle representing it only for the one instant at which that particle is one of those at the radius of inertia.

Besides finding the radial acceleration at one instant, we may require to know its mean value at all the instants throughout a certain definite time, just as in the case of tangential acceleration. It may be sufficient to take the mean as simply half the sum of the initial and final accelerations, but if this does not suffice, as many intermediate values of the acceleration as may be necessary must be found and their mean taken.

§ 27. LINEAR VELOCITY AND TOTAL ACCELERATION.

We have seen that a body might receive acceleration either in or at right angles to its direction of motion, or in both ways at once, and these two accelerations together constitute the total acceleration of the body. In general it is more convenient to treat them separately than together, because (at least in machinery) they connect themselves with totally different forces. But there is never any difficulty in adding the two accelerations together graphically if it is

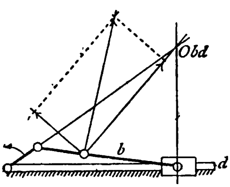


FIG. 89.

desired to find the total acceleration which a body is undergoing at any instant. It will be sufficient to give one example of this. Suppose we desire to find the total acceleration (often called the resultant acceleration) of the con-

necting rod in Fig. 89 taking quantities as on p. 190 in the last section. We have already found that the radial acceleration was 11·6 foot-seconds per second. The tangential acceleration in the position shewn, may be 6 foot-seconds per second. The total acceleration is given by the ordinary construction in Fig. 89. It is 13·1 foot-seconds per second and its direction makes an angle of about 27° with the virtual radius. An example of this kind fully worked out and commented on will be found in § 49.

§ 28. LINEAR VELOCITY AND ACCELERATION DIAGRAMS.

In several sections of Chapter V. we saw how to construct diagrams of the velocity of a point in a mechanism; it now remains to show how to connect such diagrams with a graphic representation of the acceleration occurring as the velocity changes. The particular velocity diagrams hitherto drawn were constructed on the assumption that we knew the velocity of at least some one point in a mechanism. We shall presently see how to construct such a diagram without this assumption, calculating the velocity at each instant from the forces in action. This matter is not one which need concern us just now, however. We may assume that we have, to start with, a diagram of the linear velocity of a body constructed by any method. It must be assumed here either that the body has only a motion of translation or that (if the body be rotating in any way) the diagram represents the linear velocities *in one particular* direction of a particle representing the body. Our problem shall now be :—given such a curve, its ordinates represent-

ing velocities, to construct by its aid another curve whose ordinates shall represent the corresponding accelerations. Velocity diagrams are of two types; in most we have hitherto constructed the abscissæ have represented the *distances* moved through by the point, but we might equally well construct a diagram in which abscissæ should represent *intervals of time* instead of intervals of space. As the latter type of diagram is somewhat the simpler we shall commence with it.

In Fig. 90 the equal spaces (abscissæ) marked 1, 2, 3, 4,

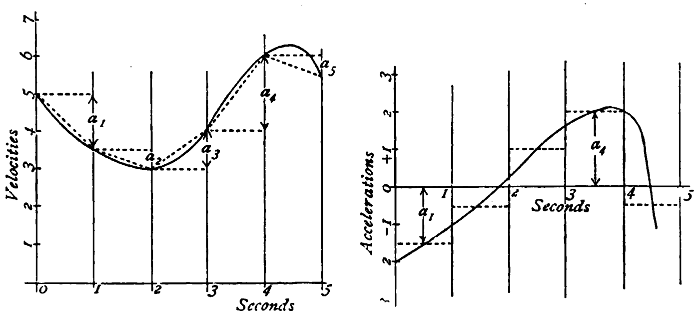


FIG. 90.

&c., on the horizontal axis, stand for equal intervals of time (seconds), while the curve above represents the corresponding velocities to the scale which is marked on the vertical axis. The body for which the diagram is drawn decreases in speed from five to three feet per second in the first two seconds, then increases in speed to six feet per second in the next two, and then falls to about five feet per second again. The acceleration must therefore be negative at first, then positive, then negative again. Taking the acceleration as constant through each interval,

i.e. treating the curve as if it were a polygon with vertices where it cuts the second lines, the acceleration during each interval is $\frac{v_2-v_1}{t}$ and as in this case we have the initial and final velocities for each single second given in the diagram, t becomes $=1$, and $a=v_2-v_1$, so that it is only necessary to measure off the differences between the first and last ordinates for each second (marked a_1, a_2, a_3 &c.), and set them off above or below any axis, and opposite the middle points of the time intervals.[1] The accelerations being thus numerically equal to the differences of velocity in each second, *the scale of accelerations will be the same as the scale of velocities*, that is the length on the paper which stands in the one case for one foot-per-second will also stand in the other for one foot-second-per-second. In this diagram a separate axis is used for setting off the accelerations, but in general, to save space, the horizontal axis of the velocity diagram will be used also as the axis for the acceleration curve. Positive acceleration will be set upwards from the axis and negative downwards.

The points thus determined in an acceleration curve correspond to *the values of the mean accelerations during each time interval on the assumption that the velocity changes uniformly during each interval*, and that therefore the acceleration is constant during each interval. The velocity curve is in fact assumed to be the dotted polygon, and the acceleration diagram the stepped (dotted) line, the acceleration changing suddenly and only at the end of each second. This is the simplest relation which can exist between velocity and acceleration curves. A constant rate of change of velocity makes the velocity curve a straight

[1] Opposite the *middle* and not the *end* points, because they represent the average acceleration during the whole of the time interval.

line of uniform slope, and the acceleration curve a straight line parallel to the axis. But in reality the rate of change of the velocity is *not* uniform during each second, but changes continually,—both velocity and acceleration therefore are represented by continuous curves. Instead of drawing the acceleration curve as a series of steps, we therefore draw a continuous line through the points which we have determined and take the ordinates of this line as an approximation to the real (continually changing) accelerations at each point.

The closer the time intervals be taken the more nearly does the velocity polygon coincide with the velocity curve, and the closer therefore does the acceleration curve approximate to a true representation of the real accelerations at each instant, and there is never any practical difficulty in the way of making the approximation quite as good as the conditions of the problem require.

But we have seen that we can obtain an absolutely exact value of the acceleration at any given instant without using any approximative polygon, or making any assumptions as to the velocity increasing uniformly over certain time intervals, however short. The method of doing this was described and proved in § 24, in connection with Fig. 86. To apply that method in a case like this it is necessary only to draw tangents to the velocity curve at the point where the acceleration is required, and take for the acceleration at the point the rise or fall of this tangent (measured on the velocity scale) in whatever distance stands for one second. The construction is shown in Fig. 91. It gives an acceleration diagram sensibly differing from the former only during the fifth second, where a return curvature in the velocity curve causes it to differ very much from the assumed polygon side.

The only reason for not using this method always in preference to the former is that although it is absolutely correct if the tangent be drawn rightly, yet in most cases we are only able to guess at the tangent. The velocity curve

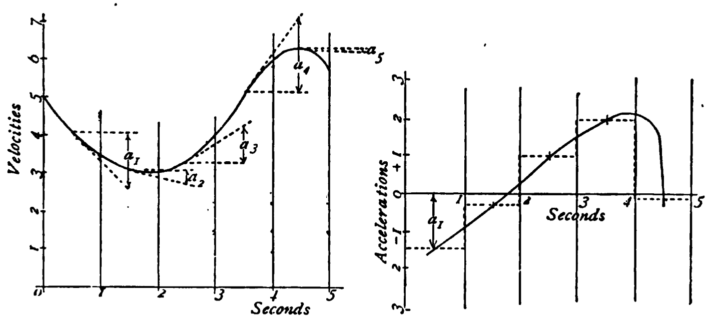


FIG. 91.

is seldom a conic or other curve whose tangent can be accurately drawn at all readily. The error introduced by wrongly drawn tangents may easily be greater than that due to the assumptions made in connection with Fig. 90. Neither errors are, however, cumulative.

In any case the tangential method should be used wherever, as in the fifth second in Fig. 90, the velocity curve differs very much from its representative polygon. Wherever, also, as at points in the second and fourth seconds above, the curve is parallel to the axis, its tangent therefore parallel to the axis, it should be remembered that there can be no acceleration, and that therefore the acceleration curve must either cut or touch the axis opposite these points.

Fig. 92 represents the case already discussed of uniform (positive) acceleration, where $v_2 - v_1$ is the same not only

for every second, but also at each instant during every second, and the acceleration curve is a straight line parallel to the axis. The scales are the same as in the last figure.

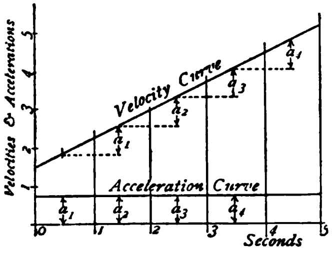


FIG. 92.

Fig. 93 represents the case of a train starting from a station and gradually attaining a maximum and constant velocity. Here the marked points on the axis represent minutes, so

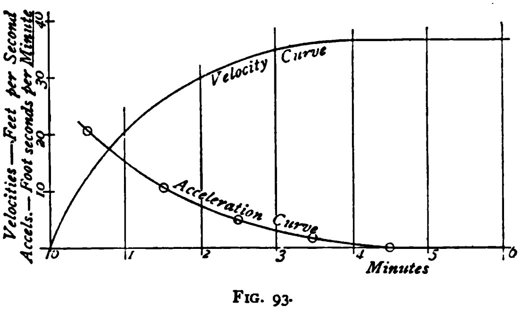


FIG. 93.

that if the accelerations be, as before, set off *equal* to the difference between the initial and final velocities for each division they are sixty times too great, as that difference

represents the change of speed in sixty seconds instead of in one. But it would be very inconvenient to divide all the differences of velocity by sixty, and quite unnecessary. To make matters right we have only to make the acceleration scale sixty times the velocity scale, as is done in the figure. This simply amounts to making the same distance stand for an acceleration of a foot-second *per minute* as stands for a velocity of a foot per second.

Precisely the same thing, of course, applies if the tangential method of Fig. 91 be used instead of the polygonal method.

In Fig. 93, there is not only one point in the velocity curve where its tangent becomes parallel to the axis, but the whole curve becomes parallel to the axis and remains so from the end of the fourth minute. The velocity in this part of the curve is therefore constant, and the acceleration zero, and we find, correspondingly, that the acceleration curve runs into the axis and disappears just when the velocity curve becomes horizontal.

The diagram Fig. 93 also affords a good illustration of a point about which it is important that there should be no confusion. The velocity of the train keeps on increasing until its maximum is reached, but the rate at which the velocity changes, *i.e.* the acceleration, keeps on *diminishing*, and becomes zero as soon as the velocity becomes constant, which happens in this case to be also when the velocity becomes a maximum. Rate of change of velocity and rate of change of acceleration are quite independent of each other, and must not be in any way confounded with one another.

If the accelerations were given in any case, and the velocities were to be found, it is obvious that the velocity curve could be set off from the acceleration curve without

any constructive difficulty, if the absolute velocity at any one point were known to start with.

Fig. 94 represents the second type of velocity curve where the horizontal abscissæ stand for equal *distances*, instead of equal *times*. The construction for the acceleration is shown in detail in two of the spaces. It is simply as follows:—through the mid point M of any chord of the velocity line draw a normal MN. The segment of

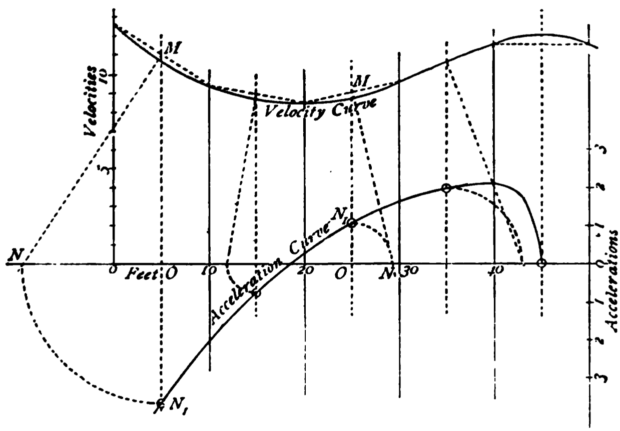


FIG. 94.

the axis ON lying under MN is the acceleration, and has only got to be turned upwards or downwards to ON_1 to give the ordinate of the acceleration curve.

The proof of this very simple construction is as follows, the letters referring to Fig. 95, and the assumptions as to uniformity of acceleration being the same as those made on p. 195. Let AA_1 and BB_1 be the initial and final velocities v_1 and v_2 respectively for a distance

s represented by AB. Then OM will be equal to the average velocity $\frac{v_2 + v_1}{2}$, and CB_1 to the difference between the velocities $v_2 - v_1$. But by equation 3, page 179, $a = \frac{v_2^2 - v_1^2}{2s}$, which may be written $a = \frac{(v_2+v_1)(v_2-v_1)}{2s}$. This may be put in the form of the proportion $\frac{a}{\frac{v_2+v_1}{2}} = \frac{v_2 - v_1}{s}$. The latter ratio is given by the sides B_1C and A_1C respectively of the triangle A_1B_1C, so that to find a we

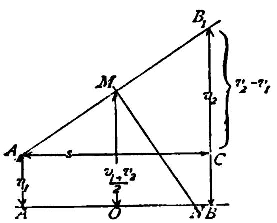


FIG. 95.

require only to draw a similar triangle to this, and make one side of it $= \frac{{}_2+v}{2}$. But by drawing MN perpendicular to A_1B_1 we obtain at once in MON such a triangle, having the side MO equal to $\frac{v_2+v_1}{2}$, and therefore the side ON must be equal to the required acceleration, and this is exactly the construction described above with Fig. 94. If the velocity scale happens to be the same as the distance scale, *i.e.* if the same length stands for one foot-per-second and for one foot, the acceleration must be read off (in foot-

seconds per second) on the same scale. If the two scales, however, be different, as will generally be the case, then if there be n times as many units of velocity in a given length as there are units of distance, the accelerations must be read on the velocity scale and multiplied by n, or, what is the same thing, read on a scale having n times as many units per inch as the velocity scale. Fig. 94 is drawn with a velocity scale of 10 feet per second per inch, and a distance scale of 20 feet per inch; n is therefore 0·5, and the acceleration scale is 5 foot-seconds per second to the inch.

In actual construction the points M have not been taken on the curve, but as midpoints of chords. This is merely for convenience of drawing, and gives an acceleration curve which corresponds really to the velocities of the dotted polygon in the figure instead of to the continuous velocity curve actually drawn. As however the vertices of that polygon are actually points on the curve, there are no cumulative errors caused in this way, but only a slight distortion of the acceleration curve, not sufficient to interfere with its usefulness for most purposes. Should the curvature of the velocity line be anywhere very sharp, it is only necessary to take points on that portion of the curve somewhat closer together than elsewhere.

But it is possible by a method analogous to that of Fig. 91, to find the actual acceleration corresponding to any point of the velocity curve without resorting to any approximative method of polygons, or assumptions as to the uniformity of the acceleration. For in the proportion

$$\frac{a}{\frac{v_2 + v_1}{2}} = \frac{v_2 - v_1}{s}$$

already looked at, a is the acceleration at a point where the velocity is $\frac{v_2 + v_1}{2}$. If we take A and B close together and both close to O, a becomes the actual acceleration at the point M on the velocity curve, and $\frac{v_2 + v_1}{2}$ becomes equal to OM. Neither $(v_2 - v_1)$ nor s can be measured separately, but the ratio between them can still be obtained. For A_1 and B_1 become consecutive points on the velocity curve, and the line joining them is the tangent to the curve at M. The ratio $\frac{v_2 - v_1}{s}$ becomes therefore simply the tangent of the angle $B_1 A_1 C$ between the tangent to the curve and the axis. **If then we draw a normal to the velocity curve at any point whatever, such as M, the "sub-normal," NO, or projection of MN upon the axis, is the acceleration at the point N.**[1] The scale on which the sub-normals are to be measured is the acceleration scale already discussed. The drawback to the use of this method is that already discussed on p. 197, that considerable errors may be introduced by a mistake in drawing the normal, while there is no method by which such mistakes can be certainly avoided. The nature of the curve compels us, in almost all cases, to guess at the tangents, and draw the normals by their help.

Fig. 96 shows the construction of Fig. 94 applied to such a case as that of a train being brought to a standstill. The initial velocity of the train is 37 feet per second, and it is brought to rest in about 600 yards. The diagram has been engraved with a velocity scale of 25 feet per second

[1] This can be proved more elegantly by the aid of the differential calculus, which would, however, be out of place here. This general method was first used, so far as I know, by Dr. Pröll, in his *Graphische Dynamik* (Leipzic, — Felix).

to the inch, and a distance scale of 250 yards or 750 feet to the inch. The value of n is therefore $\frac{1}{30}$, and the scale for acceleration $25 \times \frac{1}{30}$, or $\frac{5}{6}$ foot-seconds per second to the inch. The acceleration could either be read on the velocity

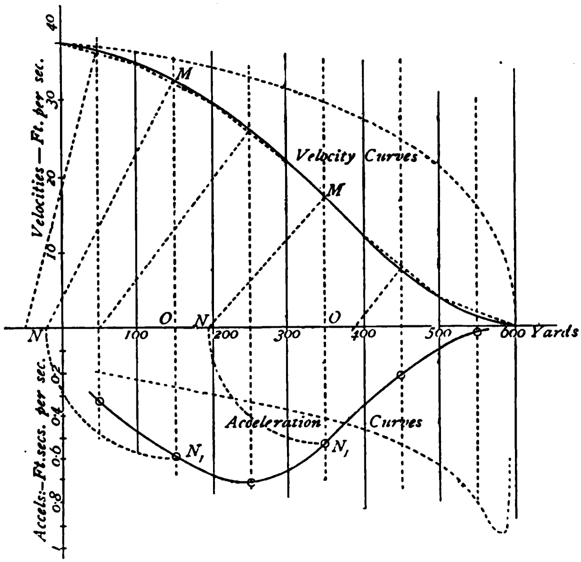


FIG. 96.

scale and divided by 30, or read on a scale of 1·2 inches per unit of acceleration.

It will be noticed that the acceleration begins slowly and gradually increases, being greatest where the velocity curve is steepest (at between 200 and 300 yards), and gradually getting less as the train comes to rest. This is not a necessary consequence of the train's velocity be-

coming slower, but depends solely on the *rate* at which the slackening of speed occurs. If, for instance, the brakes had not acted promptly, and had been put very hard on at the end, the velocity and acceleration curves might have been as dotted, when the maximum acceleration occurs almost at the end, a state of affairs very uncomfortable for the passengers.

Here, as with the former diagrams, a line of constant velocity gives a line of no acceleration. But a straight velocity diagram, as Fig. 92, no longer corresponds to uniform acceleration, for as the speed gets higher the given distance is traversed in a less and less time; and therefore if the gain of velocity in each interval of distance is the same, this gain must occupy a shorter and shorter time, so that the rate of increase of the velocity, as well as the velocity itself, must increase. This is shown graphically in Fig. 97, where the velocity increases uniformly, and the

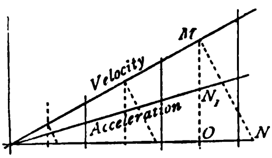


FIG. 97.

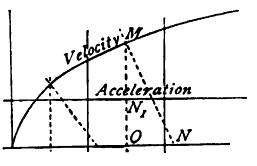


FIG. 98.

acceleration or rate of increase of the velocity increases uniformly also.

If the acceleration be constant, the velocity diagram is a parabola, the constancy of the sub-normal being a characteristic property of that curve. The case is shown in Fig. 98.

The process of finding the velocity curve from the

acceleration is not quite so simple here as in the former case, but does not present any difficulty. It is necessary of course, in both cases equally, that the initial velocity, or the velocity at some one instant, should be known to start with. Fig. 99 shows the construction, with the acceleration diagram drawn separately for clearness' sake. Let us suppose here the initial velocity to be zero, so that A will be the starting point of the velocity curve. From the mid point O of the first distance interval set off ON along the axis,

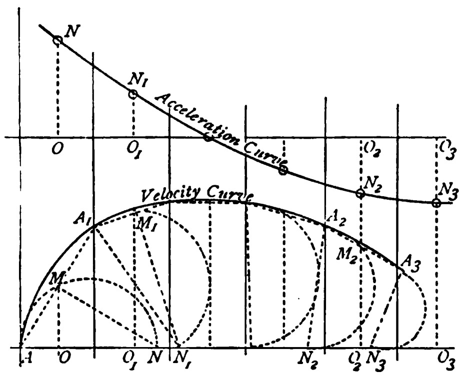


FIG. 99.

and draw a semicircle with diameter AN cutting the vertical over O in M. Then M is a point in the velocity polygon, and the line AMA_1 can be drawn as the first line in it. The proof is simply that MN is normal to AMA_1 (AMN being the angle in a semicircle), and that ON has been made by construction equal to the acceleration. For the point M_1 in the next division of the velocity diagram a similar construction is to be used, only taking O_1N_1 for the

acceleration and drawing the semicircle on A_1N_1 instead of on AN. For the third distance interval no construction is needed in this case, for the acceleration is zero (the acceleration line cutting the axis in the middle of the space), and the velocity line is therefore horizontal. For the fourth and fifth interval it is to be noticed that the acceleration is *negative*, so that O_2N_2 has to be set backward instead of forward along the axis; and the same construction will be used for O_3N_3. It is obvious from the position of the semicircle on the line A_3N_3, that we can not continue the construction over another space, for the negative acceleration has become so large that the semicircle would not cut the vertical line through O_3 at all. This does not, however, show any defect in the construction, but only indicates that the negative acceleration has been so great that *the body has commenced to move backwards* before it has passed through another distance interval, so that there is no point at all in its velocity curve vertically over O_3. In a case where it is important to find as exactly as possible the distance which will be moved through by a body under given acceleration before it comes to rest and changes its sense of motion, the construction must be made for points as close together as possible.

On p. 202 we saw that in this case if there were n units of velocity in the length standing for one unit of distance, there would be n units of acceleration in the length standing for one unit of velocity. This enabled us to find the acceleration scale when we knew the velocity and distance scales. We have now to deal with the converse problem, namely, finding the velocity scale from given acceleration and distance scales. The relation between them is simply that if we call the number of units of acceleration in the length standing for one unit of distance n^2,

the number of units of acceleration in the length standing for one unit of velocity will be n. Thus if (as in the pumping-engine problem of § 45) we find the distance scale is 2 feet per inch, and the acceleration scale 32 foot-seconds per second per inch, we have $\frac{32}{2} = 16 = n^2$, and the velocity scale must be $\frac{32}{n} = \frac{32}{4} = 8$ feet-per-second per inch.

It sometimes happens that a set of observations of velocity (say of a train) made at constant time intervals has to be reduced to distance intervals, or *vice versâ*. This is very easily done graphically, and it will be worth while to examine the construction. Let Fig. 100 represent a part of a velocity diagram with a distance base, and drawn with equal scales of velocity and distance. By the equation on p. 164, $s = \frac{v_2 + v_1}{2} t$, or $\frac{s}{\frac{v_2 + v_1}{2}} = \frac{t}{1}$. The distance O_1M_1 in the figure is equal to $\frac{v_2 + v_1}{2}$, and OO_1 is equal to one of the distance intervals, or s. If therefore any convenient distance, as O_1N_1, be taken for a time unit, and N_1T_1 drawn parallel to M_1O, then O_1T_1 will be the time taken by the body in passing through the distance interval s, at the commencement of which the velocity was v_1, and at the end of which it was v_2 A similar construction gives O_2T_2 as the time for the second distance interval, O_3T_3 as the time for the third, &c. These intervals can then be set off as t_1, t_2, t_3, &c., along the base line of a new diagram, and the velocity ordinates transferred to their proper places on it, each at the end of its time interval, as is done in the figure. If the total time corresponding to the total distance be known, it is most convenient to make the length which stands for

the time unit such as to make both the diagrams of the same length. The dotted line in the figure shows the form that would be taken by the curve in this case if this were done.

In general it may happen that it is not convenient to take the same scale for distance as for velocity. If in any such case there are n times as many units of velocity as of distance in a given length on the paper, the length O_1T_1 will be n times too great referred to O_1N_1 as unit. In such cases,

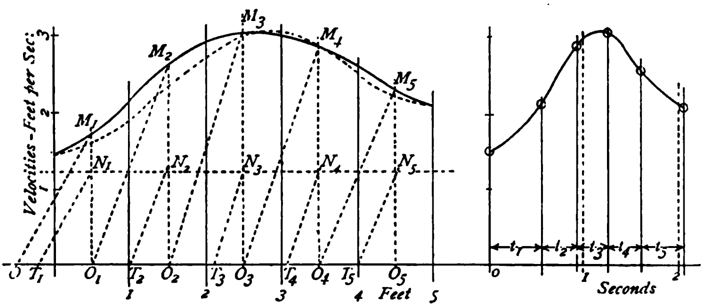


FIG. 100.

therefore, the length $O_1N_1 = O_2N_2$, &c., must be taken, not as equal to the intended time unit, but as $\frac{1}{n}$th of it. Thus if in any diagram the velocity scale be 10 feet-per-second to the inch, and the distance scale 2 feet to the inch, and the time unit be 1 inch, the distance O_1N_1 must be 0·2 inch, the ratio n being $= 5$. If this procedure makes O_1N_1 too small (or too great, in cases where n is a fraction, as on p. 204), all that is necessary is to take it any convenient fraction or multiple of the time unit, and reduce or increase all the time intervals O_1T_1, O_2T_2, &c., in the proper ratio.

The converse operation of turning a diagram constructed on a time base into one upon a distance base is much simpler. The distance passed through in any interval is equal to the mean velocity during that interval multiplied by the time, $s = \frac{v_2 + v_1}{2} t$. If therefore the time interval be unity (as in Fig. 90), *i.e.* if $t = 1$, it follows that $s = \frac{v_2 + v_1}{2}$, which we have seen to be simply the middle ordinate of the velocity curve for the given interval, as OM, O_1M_1, O_2M_2, in Fig. 101. The scale for distances

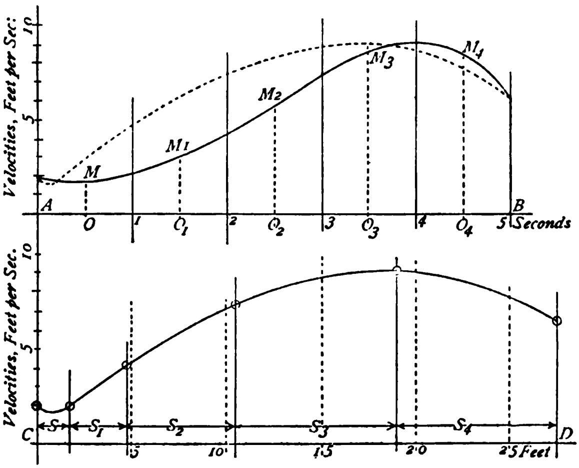


FIG. 101.

is in this case the same as the scale used for velocities, that is, the length that stands for a speed of one foot-per-second will stand also for a distance of one foot. If this turns out

to be an inconvenient scale for distances, it is only necessary to measure the middle ordinates on the scale just mentioned, and set them off on any scale that is more convenient. In this case, as in the last, it may be convenient wherever the total time and the total distance are both known, to make the scale for the latter such as shall make the length of the two diagrams equal, or in any case to reduce the distance base, after the new curve is drawn, to a length equal to the time base. This has been done in the figure, and the dotted curve is the result, which shows how essentially different the same velocities may look according to whether the abscissæ are times or distances. It is obvious that the whole length of the distance base is equal to

$$OM + O_1M_1 + O_2M_2 + \ . \ . \ . \ = CD.$$

This length can be found without drawing the second curve at all. The different values of OM, O_1M_1, &c., can then each be reduced in the ratio $\frac{AB}{CD}$, and set off consecutively from A along the original base, and the dotted curve constructed at once. In this case, of course, the length of the distance unit will be only $\frac{AB}{CD}$ of the unit of velocity, and the fraction will not probably be a convenient one.

§ 29.—ANGULAR ACCELERATION.

The angular velocity of a body may receive tangential accelerations exactly corresponding to those which we have already examined in connection with its linear velocity. As angular velocity is measured in angular-units-per-second, angular acceleration will be measured in angular-units-per-

second per second, or let us say for shortness' sake, angle-seconds per second, just as we before said foot-seconds per second. We have already seen that the angular velocity of a body is numerically equal to the linear velocity of a point in the body at unit radius in feet per second. So the angular acceleration of a body in angle-seconds per second must be numerically equal to the linear acceleration of a point in the body at unit radius in foot-seconds per second. If therefore the radius of inertia of a body be r feet and the linear and angular acceleration of the body be a and a_a respectively, there is between these linear and angular accelerations which the body is undergoing at any one particular instant the simple relation

$$\frac{a}{r} = a_a$$
$$a = a_a r,$$

a and a_a being expressed in the units which we have just mentioned, and r in feet.

If the body is turning about a permanent centre (as a fly-wheel, for instance), the virtual radius of every point in it remains unchanged. The rate of change of angular velocity, *i.e.* the angular acceleration, is the same for every point in the body, for at any instant all points in the body have the same angular velocity. The linear velocity of each point therefore changes at the same rate as the angular velocity of the body.

If, on the other hand, the point about which the body is turning varies continually (as when a body is moving about a series of virtual centres in succession), the virtual radius of each point also changes continually. Although, therefore, the angular acceleration at any instant must still be the same for all points in the body, the linear velocity of any particular

point may be changing at a rate totally different from the rate of change of the angular velocity. Take, for example, the case of a railway-wagon wheel when the train is moving with uniform velocity. The virtual centre of the wheel is always the point which touches the rail (see p. 149). About this particular point in itself the wheel as a whole is at each instant turning with uniform angular velocity, and therefore with angular acceleration = 0. But the different points of the wheel are continually changing their velocities as the motion of the wagon causes them to alter their distances from the virtual centre. Hence these different points have at any instant very different linear accelerations. In this instance, however, the linear acceleration as well as the angular acceleration of the wheel *as a whole* is zero, for (assuming the wheel to be a disc-wheel of any type) the radius of inertia remains always constant, although the points in the body which lie at that radius are changing continually. Hence a_a and r being both constant, the product $a_a r$, which we have just seen to be equal to a, is constant also.

If, on the other hand, r changes continually, as in the motion of a connecting rod, for example, it is possible, not only that the linear acceleration of different points shall be very different at any one time, but also that while both linear and angular accelerations vary, they will vary at very different rates, or that one might vary while the other remained constant. But at any one instant the one can be found from the other if only the virtual centre and the radius of inertia of the body be known; and from either of them, with similar data, can be found the acceleration of any special point in the body that may be required.

In all cases, except the two special ones to be examined

immediately, it is thus purely a matter of convenience whether we deal with the linear or the angular velocities or accelerations of a body, and we choose between them according to the nature of the problem to be solved. The one can be at once converted into the other by a simple numerical operation. But it must not be forgotten that the linear velocity of a rotating body is only the sum of the components of the velocities of its different points *in one direction.* The radial acceleration of a body also can be stated in angular units as well as in linear ones, for $v = v_a r$, so that the radial acceleration $\frac{v^2}{r} = \frac{v_a{}^2 r^2}{r} = v_a{}^2 r$, and this form can be used equally with the other if it happens to be more convenient.

There is, however, one special case in which we can measure the velocity of a body only in linear units, and another in which we can express it only in angular units. The former occurs when the motion of the body is a simple translation, or rotation about a point at infinity. The linear acceleration may have any finite value a, but r is infinitely great, so that a_a, which is $= \frac{a}{r}$, must be $= \frac{a}{\infty}$ that it must be infinitely small. This corresponds to what we know to be the case, that a body having a simple translation does not turn through any angle at all, so that the number of angular units through which it moves in any time whatever (so long as the motion continues a simple translation) is always $= 0$. Under these same circumstances the radial acceleration is also equal to 0, and for the same reason, for if $r = \infty$, $\frac{v^2}{r}$ must be always equal to 0, so long as v has any finite value. Hence, **when a body has a motion of simple translation, its velocity and acceleration**

can only be measured in linear units, and its radial acceleration is zero.

When a body, on the other hand, is revolving about its own mass-centre, that centre becomes a fixed point, and has no velocity. But the velocity of the mass-centre of any body is, by definition, the mean of the velocities of all its points. The mean velocity of all the points of such a body in any direction is therefore zero—it has **no linear velocity.** But the body has angular velocity, for the angle through which it turns as a whole does not depend on the point about which it is turning, and can be just as easily measured here as under any other circumstances. The acceleration of the body can also, for similar reasons, be measured in angular units. The body has no radial acceleration **as a whole,** the radial accelerations of its different points exactly balance each other. This is familiar enough experimentally; a well-balanced top, for instance, spins steadily on a smooth surface when left to itself, without any tendency to move off in any one direction, the tangential tendencies of its different particles in different directions exactly balancing each other. We have already seen (p. 190) that in a case like this, although the body as a whole has no radial acceleration, all its component parts have none the less their individual accelerations, which under certain circumstances require careful consideration. Summing up, we may say **when a body has a motion of simple rotation about its own mass-centre, its velocity and acceleration can only be measured in angular units, and its radial acceleration as a whole is zero.**

The problems connected with acceleration in angular velocity being so similar to the corresponding problems of §§ 24 and 25, it is not necessary to give more than

one or two examples of them. In these we shall use the letters v_a, s_a and a_a, to denote respectively angular velocity, distance and acceleration, all three of course measured in the angular units already discussed.

The difference between the acceleration of a body at one instant and the mean value of its acceleration at a number of successive instants, has here again to be kept in mind, and the remarks made on this point and as to the mode of obtaining a mean, on p. 171, apply equally here as before, and need not be repeated.

A body has its angular velocity increased at a uniform rate from 5 to 15 during 14 complete revolutions. What time does the change of speed occupy?

Here the mean angular velocity $\frac{v_{a_2} + v_{a_1}}{2} = 10$, and 14 revolutions $= 14 \times 2\pi$ or 88 (nearly) units of angular motion, so that by equation 1, p. 179, $t = \frac{88}{10} = 8 \cdot 8$ seconds.

A wheel revolving at 420 revolutions per minute receives a uniform angular acceleration of $-1 \cdot 5$ during a minute and a half. What will be its speed at the end of this time?

420 revolutions per minute is equal to an angular velocity of $\frac{420}{60} \times 2\pi$ or 44 units nearly. From equation 2, p. 173, the final velocity is $v_{a_2} = v_{a_1} + a_a t = 44 - (1 \cdot 5 \times 90) = -91$, and this retransformed into revolutions per minute is $-\left(\frac{91 \times 60}{2\pi}\right) = -868$ revolutions, the negative sign indicating that the body is turning in the opposite sense to that of its original motion.

A fly-wheel is making 77 revolutions per minute when the driving force ceases to act, and the frictional resistances cause it to undergo a uniform negative acceleration of 0·08. How long will it be in coming to rest?

Here $v_{a_2} = 0$, while 77 revolutions per minute correspond to an angular velocity of $\frac{77}{60} \times 2\pi = 8$ nearly. From equation 12, p. 174,

$$t = -\frac{v_a}{a_a} = \frac{8}{-0 \cdot 08} = 100 \text{ seconds.}$$

§ 30.—FORCE, MASS, AND WEIGHT.

It is sufficient for our purposes to define force, as we have already done, as the cause of acceleration.[1] Of force in the abstract we know nothing, but we find that we can always measure and compare forces by measuring and comparing the accelerations they can produce, (see also p. 230).

We may define *equal* forces to be those which can give equal accelerations to the same body—which can, in other words, cause the same change of velocity in the same body after acting upon it for the same interval of time.

But we have, of course, to do with an infinity of different bodies, and we find that in general equal forces can *not* cause equal accelerations in these bodies. We must therefore find some way of comparing the forces which we find to produce the same acceleration in different bodies, and further to compare those which produce different accelerations in different bodies.

Moreover, we express forces, for most ordinary purposes, in units (pounds) not directly connected with acceleration or motion in any way, and we must find how this common standard of force is related to the true standard which is derived directly from the connection of force with acceleration.

We know from observation that if we have any number of different bodies at the same place, and allow all to fall freely under what we call the force of the earth's attraction, all will receive the same acceleration. But we know further

[1] The word acceleration is here used in its most general sense, but in all parts of this section where numerical quantities are connected with accelerations it is to be understood that *linear* accelerations are meant, unless the contrary be stated expressly. The relations of force to angular accelerations will be considered separately in § 32.

that the attraction of the earth to any body, or the "attraction of gravitation," is measured by the *weight* of that body. The bodies in question are therefore acted on by forces proportional to their own weights, and these forces produce upon all of them *the same acceleration.* Hence with different bodies, so long as they are at the same place upon the earth's surface, the forces necessary to give them this particular acceleration are proportional to their weights. Measuring the magnitude of forces only by the magnitudes of the accelerations they can produce, we assume that for other accelerations the two quantities are directly proportional—that a doubled acceleration, for instance, requires (other things being equal) a doubled force, and so on.

If, then, we were to take for our unit of force the force necessary to give unit (linear) acceleration to a body of unit weight, we should have the magnitude of the force causing a given acceleration in a body of given weight proportional to the (number of units of) weight of the body and to the (number of units of) acceleration received by it. If we write f for the force causing an acceleration a in a body of weight w, we could write, algebraically, $f = wa$, or $\frac{f}{wa} = \text{constant} = 1$. A force 12, for example, would give to a body weighing 2 pounds an acceleration of 6 foot-seconds per second, to a body weighing 4 pounds an acceleration of 3 foot-seconds per second, and so on.

By this plan we should be able to compare forces with each other *if the bodies on which they acted were all at the same place.* This limitation is not of itself of any practical importance in problems connected with machinery, but it must nevertheless be got rid of if we are to obtain any accurate standard of force. For we have defined those forces to be equal which give equal accelerations to the same body

(without any conditions as to the position of the body), and this is incompatible with the formula just given. For if a force f give to a body of weight w, *at the sea level*, an acceleration $= a$, then if the same body be taken to the top of a mountain, and the same force caused to act upon it, it would according to the formula of the last paragraph produce, not the same, but a *greater* acceleration. For $a = \frac{f}{w}$, and by hypothesis f remains the same, while we know that under the circumstances mentioned w, the weight of the body, would *diminish*. The value of f divided by w would therefore *increase*.

Although the weight of a body changes with its position, what is usually called the *quantity of matter* in the body, or, shortly, its **mass**, does not change. What is required, therefore, is that we should find some means of measuring this mass, or unchangeable quantity. Let us indicate the acceleration produced by the action of gravity upon a freely falling body by the symbol g, so that we may say that in every second during which it acts gravity gives to such a body an additional velocity of g feet per second. If the position of the body be changed, its weight alters from w, say, to w' or w'', and the acceleration produced by gravity alters also, say to g' or g'' respectively. But this acceleration is found by experiment to vary in exactly the same proportion as the weight, so that for one and the same body the ratio which the value of g bears to that of w is constant. In symbols $\frac{w}{g} = \frac{w'}{g'} = \frac{w''}{g''} = \ldots$ &c. Here, then, we have a number which has always the same value for the same body independently of its position in space. **This number is therefore taken to represent the mass of the body, or so-called "quantity of matter" contained in it.**

To obtain a standard for the comparison of forces which shall agree with our definition of equal forces, we have only to substitute unit of mass for unit of weight in the last definition, and take for our unit of force **the force which can give unit acceleration to a body of unit mass.** In symbols $f = \frac{w}{g} a, = \frac{w}{g} \cdot \frac{(v_2 - v_1)}{t}$, or writing m for the number of units of mass in the body $f = ma = m \frac{(v_2 - v_1)}{t}$, a being, as before, the *acceleration*, or the rate at which the velocity can be caused to increase by the action of the force f. The mass m being constant for the same body under all circumstances, forces which are equal according to this definition always produce equal accelerations in the same body. By moving it into different positions on the earth's surface we only change $\frac{w}{g}$ into $\frac{w'}{g'}, \frac{w''}{g''}$, and so on, but all of these are equal to m.

For most ordinary purposes the acceleration g may be taken as constant and as numerically equal to 32 (more closely 32·2) foot-seconds per second. By expressing weights, then, in pounds, and taking $g = 32$, we see that the mass of a body whose weight is 1 pound is $\frac{1}{32}$, a body weighing 32 pounds having a mass $\frac{32}{32}$, or unity. The unit of mass is therefore in round numbers 32 times the unit of weight. The unit force is therefore the force required to produce unit acceleration in a body containing (about) 32 units of weight, or (taking the usual standards) to give in one second an additional velocity of one foot per second to a body weighing (about) 32 pounds.

But the equation just given, $f = ma$, enables us to compare forces from a somewhat different point of view, and one considerably more easy to realise. If, namely, the

acceleration produced in a body by any force be equal to the "acceleration due to gravity," or g— foot-seconds per second, then the force necessary to produce that change will be $f = mg$ units, and this is equal to $\frac{w}{g}.g$ or simply w units. That is to say, **the number of units of force necessary to give to any body an acceleration equal to that which gravity can produce on the same body at the same place is numerically equal to the number of units of weight in the body.** This long statement is often shortened by saying simply that the force necessary to produce this particular acceleration is *equal to the weight of the body*, but the shorter statement is inaccurate and very apt to mislead. The coincidence between force and weight is merely a numerical one,—in a special case the number of units of force necessary to produce a certain acceleration is equal to the number of units of weight in the body accelerated. It does not follow that a unit of force is *equal* to a unit of weight, and that the two should go by the same name, or even that they should be commensurable quantities at all. If, for instance, a material be selling at twenty shillings a ton, the number of shillings paid for any quantity of it is numerically equal to the number of hundred-weights of material bought. Taking shillings and hundred-weights as units, the number of units of price would be numerically equal to the number of units of material. But we do not say that the price paid is *equal* to the weight purchased, nor do we give the same name to each of the two different units. This, however, is just what *has* been done in the case of force and weight. The unit of force *does* commonly receive the same name as the unit of weight, and we talk of a force of so many pounds or tons, just as we do of a weight of so many pounds or

tons. It is unfortunate that this has been done, for the consequence of it has been a frequent confusion between the two, and naturally, a misapprehension of the relations between the things themselves. At present, however, the custom of calling the unit of force a pound is so universal in problems bearing upon practical work, that any radical change would probably have drawbacks in relation to these problems more than sufficient to balance its advantages. Choosing the less of two evils, therefore, we are compelled to retain the word pound for both unit of force and unit of weight. The student must most distinctly remember, however, that this is merely an identity in name and not in nature. A pound weight is no more identical with a pound of force, than it is with a pound sterling. It need not be more difficult to keep weight and force distinct than to distinguish weight and money. In any cases where it is specially desirable to emphasise the distinction we may speak of weight-pounds and force-pounds, just as we speak of pounds weight and pounds sterling under similar circumstances.

Our equation for force, $f=\frac{w}{g}a$, gives us a most important proportion, of which we shall frequently have occasion to make use, viz. :—

$$\frac{f}{w}=\frac{a}{g}.$$

Put into words this is: **the (number of units of) force necessary to give to a body any acceleration a bears the same ratio to the (number of units of) weight in the body that that acceleration bears to the acceleration g due to gravity.** If, for instance, a body weighing ten pounds receive an acceleration of sixty-

four foot-seconds per second, the force causing that acceleration must have been equal to $10 \times \frac{64}{32}$ or twenty force-pounds. Had the acceleration been sixteen instead of sixty-four, the force would have been $10 \times \frac{16}{32}$ or five force-pounds. In the case, as we have seen, where the acceleration received is equal to that which would be caused by gravity on a freely falling body, the force would be $10 \times \frac{32}{32}$ or ten force-pounds, the same number that is, as the body contains pounds of weight.

Examples: What force is required to give an acceleration of 7·5 to a body whose mass is 16 units?

$f = ma = 7{\cdot}5 \times 16 = 120$ (force) pounds.

What force is required to give an acceleration of 8 to a body weighing 80 pounds?

Here we write $f = \frac{w}{g}a = \frac{80}{32} \times 8 = 20$ (force) pounds.

A body weighing 120 pounds moving at the rate of 18 feet per second has its speed increased at a uniform rate to 36 feet per second in 4·5 seconds. What force must have acted upon it to produce this acceleration?

The acceleration itself must first be found; it is $\frac{36 - 18}{4{\cdot}5} = 4$ foot-seconds per second. The force is then $\frac{120}{32} \times 4 = 15$ pounds.

A body of mass 48 is acted on by a force of 36 pounds. What acceleration will be produced in it?

From $f = ma$ we have of course $a = \frac{f}{m}$, so that here $a = \frac{36}{48} = 0{\cdot}75$ foot-seconds per second.

A body which weighs 80 pounds moves with a velocity of 5 feet per second. Its speed is gradually increased to 50 feet per second by the continued action of a force of 9 pounds. For how long a time must this force act to produce the change?

Here the unknown quantity is t in the equation $f = \frac{w}{g} \frac{(v_2 - v_1)}{t}$, from which t is equal to $\frac{w(v_2 - v_1)}{g \cdot f} = \frac{80(50 - 5)}{32 \times 9} = 12 \cdot 5$ seconds.

If the same force acted on the same body for 25 seconds what would be the final velocity of the body?

Here v_2 is the quantity required, and from the equation just written down $v_2 = \frac{g \cdot t \cdot f}{w} + v_1 = \frac{32 \times 25 \times 9}{80} + 5 = 95$ feet per second.

What has been the acceleration in the two last cases?

In the one it was $\frac{50 - 5}{12 \cdot 5} = 3 \cdot 6$ foot-seconds per second, in the other it was $\frac{95 - 5}{25} = 3 \cdot 6$ foot-seconds per second also. This calculation forms a check on the working in the last two questions, because it is evident that as the same force (9 pounds) was acting on the same body (80 pounds) in the two cases, the acceleration ought to be the same in each.

A body originally at rest is acted on by a force which gives it a velocity of 30 feet per second in 7·5 seconds. The body weighs 112 pounds: what must be the magnitude of the force?

Here the acceleration is $\frac{v_2}{t}$ (v_1 being $= 0$) or $\frac{30}{7 \cdot 5} = 4$ foot-seconds per second, and $f = \frac{w}{g} a = \frac{112}{32} \times 4 = 14$ pounds.

A body of the same weight moving with a velocity of 48 feet per second has to be brought to rest in 6 seconds: what force will do this?

Here $v_2 = 0$ and $a = -\frac{v_1}{t} = -\frac{48}{6} = -8$. The force is therefore $\frac{112}{32} \times -8 = -28$ pounds. The negative sign here shows that the sense of the force must be the same as the sense of the acceleration, the opposite sense, namely, to that of the original velocity. This holds true in all cases, the force producing any acceleration must be of the same direction and sense as the acceleration produced.

If, in the last question, the time allowed for bringing the body to rest had been reduced from six seconds to one, the acceleration would have been six times as great, and the

force required would have been increased in the same proportion. This holds good always, **every shortening of the time of the operation is accompanied by an exactly proportionate increase in the force necessary to carry it out.** If the acceleration occupied only $\frac{1}{100}$ of a second, the force must have been increased in the ratio of 6 to $\frac{1}{100}$, *i.e.*, it must have been made 600 times as great as before, or 16,800 pounds instead of 28. This shows us what the answer must be to a question sometimes asked, viz. what force could bring a body to rest instantaneously. *No* finite force could do this, but only an *infinitely great one*, and therefore we may say that it is impossible that any moving body should be instantaneously brought to rest. If the retarding force be very great, the time required for the stoppage may be very small, but *no force whatever, short of an infinitely great one, can reduce the time to absolutely zero, that is, can make the stoppage absolutely instantaneous.*

Examples.—A body weighing 10 lbs. starting from rest moves downwards for 3 seconds, at the end of which it has acquired a velocity of 96 feet per second. What has been the accelerating force?

The acceleration $a = \frac{v_2}{t} = \frac{96}{3} = 32$ or g. The body must therefore have been falling freely under gravity, the accelerating force having been simply equal to its own weight.

Weights of 10 and 12 pounds respectively are hung from the two ends of a cord over a (frictionless) pulley. With what acceleration will the heavier side descend?

The accelerating force is here $12 - 10 = 2$ force-pounds, and the mass to be accelerated is $12 + 10 = 22$. Hence

$$a = \frac{f}{m} = \frac{g.f}{w} = \frac{2 \times 32}{22} = 3 \text{ foot-seconds per second, nearly.}$$

This is a case often occurring in Cornish and other engines, and has been aptly termed a *dilution of gravity* by Dr. Lodge, because by balancing a part of the weight we increase the mass and diminish the accelerating force at the same time.

The following is an example of the same kind in a form in which it

might occur in practice:—A Cornish pumping engine has a cylinder 60 inches in diameter, and for 3 feet of its down stroke the pressure on its piston is 32 pounds per square inch. The weight of the unbalanced pit-work (pump rods, &c.) is equivalent to 20 pounds on the square inch. There is also a weight of $11\frac{1}{2}$ tons in the pit-work, which is balanced, while the weight of the beam itself, which may also be taken as balanced, is 8 tons. At what speed will the piston be moving when it has travelled 3 feet of its stroke?

It will be convenient in the first place to reduce the balanced weights to their equivalents per square inch of piston area. $\dfrac{8 \times 2240}{\pi \times 30^2} = 6{\cdot}3$ lb. per square inch, is the equivalent of the weight of the beam, and $\dfrac{11{\cdot}5 \times 2 \times 2240}{\pi \times 30^2} = 18{\cdot}2$ lbs. per square inch, is the equivalent of the balanced part of the pit work together with the (assumed equal) balance weights. To this is to be added 20 lbs. per square inch, the equivalent of the actual load to be lifted, making a total per square inch of piston of $6{\cdot}3 + 18{\cdot}2 + 20 = 44{\cdot}5$ pounds. The accelerating force is $32 - 20 = 12$ pounds, also per square inch of piston, and this force acts uniformly through a distance of 3 feet. The mass accelerated is $\dfrac{44{\cdot}5}{32} = 1{\cdot}4$ per square inch of piston, and the acceleration is therefore $a = \dfrac{f}{m} = \dfrac{12}{1{\cdot}4} = 8{\cdot}6$ foot-seconds per second. To find the velocity, equation 9 of § 25 can be used, viz. $v = \sqrt{2as}$

$$v = \sqrt{2 \times 8{\cdot}6 \times 3} = \sqrt{51{\cdot}6} = 7{\cdot}2 \text{ feet per second.}$$

If the weights here had been "undiluted," if for instance the weight of 20 lbs. per square inch had simply been hanging from the piston rod (as in a Bull engine) and there had been no balanced weights, the result would have been very different. The mass accelerated would have been $\dfrac{20}{32} = 0{\cdot}62$ per square inch of piston, and the acceleration $\dfrac{12}{0{\cdot}62} = 19$ foot-seconds per second. The velocity at the end of three feet would therefore have been $\sqrt{2 \times 19 \times 3} = \sqrt{114} = 10{\cdot}7$ feet per second.

If the acceleration to be dealt with is radial instead of tangential, it is to be handled in precisely the same way.

Suppose for example the ball whose centripetal acceleration was found on p. 190 to be of cast-iron and that its diameter is three inches. Let it be required to find the force necessary to produce the radial acceleration. The ball will weigh 3·7 lbs., and the radial force will again be $f = ma$ or $f = \frac{w}{g} a$, while we have already found that $a = \frac{v^2}{r}$. Hence $f = \frac{3·7}{32} \times \frac{5·23^2}{2·5} = 1·27$ pounds. If we placed a spring balance between the axis and the ball instead of the cord this is the tension which it would indicate for us. If the cord is too weak to withstand a pull of 1·27 pounds it will break, and the ball will fly off tangentially.

We have called the acceleration in such a case as this *centripetal*, because it is always directed towards the centre. The force causing it therefore is a *centripetal force*. But as action and reaction are equal, the force exerted by the cord on the ball and towards the centre must be exactly equalled by a force exerted by the ball on the cord and away from the centre, and therefore suitably called a *centrifugal* force. Although it is not this force but the former one which is the most directly obvious to us, and although for all purposes of calculation the one is just as suitable as the other, yet in practice it almost always happens that it is the centrifugal rather than the centripetal force which is spoken of. This would be a matter of indifference were it not that the use of this particular word has given rise to the idea that the ball or other body tends of itself to move away radially from its centre, whereas its tendency if left to itself is always to continue in its existing path, *i.e.*, to move away tangentially to its former orbit. Bearing this in mind we may use, without being misunderstood, the common expression "centrifugal force," meaning thereby simply a force equal and opposite to the centripetal force, which is the cause of centripetal or radial acceleration in a rotating body.

If, then, a body is moving with a linear velocity of v feet per second about a centre (permanent or virtual) at radius r feet, it is undergoing a radial acceleration of $\frac{v^2}{r}$ foot-seconds per second, and the centrifugal force corresponding to this acceleration will be $\frac{v^2}{r}$ pounds per unit of mass, $\frac{v^2}{gr}$ pounds per unit of weight (pound), or, in general, for a body weighing w pounds, $\frac{wv^2}{gr}$.[1]

We need only take one more example:—What is the "centrifugal force" of the connecting rod of which the particulars are given on p. 190, if its weight is 3 cwt.?

Here $w = 336$ lbs., $v = 8$ ft. per second, and $r = 5{\cdot}5$ ft. The answer is therefore

$$\frac{336 \times 8^2}{32 \times 5{\cdot}5} = 122{\cdot}2 \text{ pounds.}$$

The force which we call the weight of a body may be looked upon as the sum or resultant of all the weights of an infinite number of small particles of which the body consists. These weights together form a system of parallel forces, and their resultant, the whole weight of the body, must be in magnitude equal to their sum and in direction parallel to them, and must have some position among them dependent on the form of the body. The position of this resultant relatively to the body will be different for every different position of the body; that is, for every different position of the body it

[1] Here v is the linear velocity at right angles to its virtual radius, not of *any* point in the body, but of a point so placed that the whole mass of the body might be concentrated at it in one particle without alteration to any of the conditions. This point must clearly have the property that its linear velocity v in any direction must be the mean of the linear velocities of all the other points in the body. It will therefore be the *mass centre* of the body (p. 229).

will traverse a different set of its points. But it can be shown mathematically that for any rigid body, whether homogeneous or not, there is one point which is common to all possible positions of this resultant, which therefore form a sheaf of lines all passing through one point. This point is called the **centre of gravity** or the **mass-centre** of the body; it has lately been called[1] the **centroid** of the body. So far as the action of gravity is concerned the whole body might be replaced by a single particle, equal to it in weight, and placed in the position of its centre of gravity. We shall make use later on, as we may require them, of some of the more important properties which are deducible from the definition of this point just given, see also p. 239, etc.

§ 31. MOMENTUM AND IMPULSE. MOMENT OF INERTIA OF A PARTICLE.

A conception which is of considerable importance in Mechanics is the *quantity of motion* possessed by a body,[2] which is called the **momentum** of the body. We take it that the quantity of motion which any given body has is proportional to its mass and to the velocity with which it is moving, and take for unit of momentum (to which no particular name has been given) the quantity of motion of a body of unit mass moving at the rate of one foot per second. For any other body of mass m moving at the rate of v feet per second we have momentum $= mv = \frac{wv}{g}$. We have already

[1] See Minchin's *Treatise on Statics.*

[2] The reasoning in this section assumes that all the mass of the body is concentrated in one particle, which can have a definite linear velocity and a definite radius.

seen (p. 220) that if a force f give to a body of mass m an acceleration of a foot-seconds per second, the relation between the three quantities is $f = ma$. But if in such a case the gain of velocity $(v_2 - v_1)$ be v in t seconds, the acceleration $a = \frac{v}{t}$, so that $f = \frac{mv}{t}$. From this we get two or three important conclusions. First we have the equality $ft = mv$; which, put into words, is that **the product of a force into the time during which it acts is numerically equal to the product of the mass of the body acted on into the velocity gained during that time,** all four quantities being expressed in the proper units. The product ft is called the *impulse*, and the equation may therefore be stated simply: **the quantity of motion or momentum received by any body is numerically equal to the impulse which has caused that momentum.**

Secondly, noting that mv above does not stand for the whole momentum possessed by a body, but only for the (algebraical) increase of the momentum in a given time, we may put the equation $f = \frac{mv}{t}$ into words by saying that **force is rate of change of momentum with time.** If we know that a given body has had its quantity of motion, or momentum, increased by a certain amount during a definite time, we are therefore in a position to find the force which must have caused that change. The relation between force and momentum is thus exactly the same as that existing between acceleration and velocity, and between velocity and distance, as is seen at once on comparing the equations $v = \frac{s}{t}$, $a = \frac{v}{t}$, $f = \frac{mv}{t}$. This result has been arrived at merely by the substitution of $\frac{v}{t}$ for a in our fundamental

equation for force, and there is therefore nothing new in it. But the new light in which it presents force to us is worth while noticing, and is of much importance in the study of dynamics higher than our present subject includes.

Thirdly, passing again to the first form of our equation, $ft = mv$, we see that **a force cannot increase or diminish the momentum of a body unless it act for some finite length of time,** and that **the force necessary to produce any given change of momentum is inversely proportional to the time occupied by the change.** This we have already found in a different fashion in § 30, where some illustrations bearing on this statement are given. We need not therefore say more about it here.

Just as we can measure velocity in either linear or angular units, so, of course, we can measure momentum in corresponding fashion. The linear momentum of a body is proportional to its mass and linear velocity only, but if the body be rotating about a point, its (quantity of) angular motion, which is its *angular momentum*, will be proportional not only to its mass and linear velocity, but also to a distance or radius—which we may call the *Radius of Inertia*—which is the radius at which we might suppose (see below, p. 243) its mass to be concentrated into one particle. The unit of angular momentum, or standard by which we measure quantity of angular motion, is the motion possessed by a body of unit mass rotating with unit linear velocity about a point at unit radius. For any other mass, velocity and radius, the angular momentum of a body is mvr, or as $v = v_a r$ (see p. 167) the angular momentum might be written $mv_a r^2$. Our original equation will change to $ftr = mvr = mv_a r^2$.

The product of any quantity which has direction (such as

the impulse ft, or the momentum mv) into another quantity at right angles to it (as the radius r), is called a **moment.** Hence our equation now tells us that **the moment of the impulse is equal to the moment of the momentum caused by it.**

Further, the product of any quantity which has direction into the *square* of any other quantity at right angles to it is called a *second moment.* Using this expression, our equation tells us that **the moment of the impulse about the virtual centre of the body on which it acts is equal to the second moment of the mass of the body about the same point** (mr^2) **multiplied by the angular velocity caused by it.** This second moment, the product of a mass and the square of its virtual radius, is what is generally called the **moment of Inertia** of the body, and is indicated by the letter I. Hence we may now finally write our equation $ftr = v_a I$. The mode of determining I, as well as some of the very important cases in which it occurs, will be considered in the next section.

We have called the quantity on the left-hand side of this equation the moment of the impulse, or $ft \times r$. But it can be analysed in another fashion, which is one of more practical importance to us, namely, as $fr \times t$. Here fr is the product of a force, which has direction, into a distance or radius measured at right angles to that direction; it is in fact the moment of the force itself about the virtual centre. Hence **any change of angular momentum is equal to the moment, about the virtual centre, of the force causing it, multiplied by the time during which that moment continues.** The moment of a force about a point or an axis, or the product of the force into its perpendicular distance from that point or axis, is generally called a *static moment*, and the unit of static

moment is the moment of a force of one pound acting at a radius of one foot, which is called a foot-pound, but must not be confounded with the unit of work (§ 33) which bears the same name. The moment of any force of f pounds acting at a radius of r feet, is simply fr foot-pounds.

The same moment of force, acting on the same body for the same time, will produce the same change of momentum. But this moment may be caused by very different forces, for obviously a small force at a large radius may give the same static moment as a large force at a small radius. It is therefore not necessary for us, in the equation $frt = v_a I$, to take r as the radius of inertia (see p. 243), and to assume that the force acts at that radius. The force may act at any radius so long as the product of force and radius—the static moment—has the required value. In the equation r has simply to be the radius at which the force f acts.

Whenever we speak of momentum alone without any qualifying adjective, *linear* momentum,—the simple product mv,—must be understood, and not angular momentum. In this sense the momentum of a moving body is a sufficient measure of its whole motion only in the case where that motion is a simple translation—where all its particles move with the same velocity in the same direction. In the more usual case, where the motion of a body, here supposed to be concentrated in one particle, is a rotation, we have to measure its quantity of motion by its *angular momentum*, the product of its mass, linear velocity, and virtual radius, mvr; or of its mass, angular velocity and virtual radius squared, $mv_a r^2$. The second of these two equal expressions is of more general application than the first. Both may be made to apply to every case of rotation except that of a body turning about its own centre of gravity, but that exception is a very important one, and in it the expression

mvr cannot be used. For in such a case the body has no linear velocity v—it does not shift its position as a whole—while at the same time r is not the radius of any particular point. Here, then, we can make no use of the expression *mvr*. But the body has none the less a definite angular velocity v_a, and we shall see in § 32 that its second moment or moment of inertia, mr^2, relatively to the point about which it is rotating, must also always have a finite and determinate value. So that $v_a mr^2$, or $v_a I$, the angular momentum of a body, can always be found, and must always, so long as the body has any motion at all, have a finite value.

Examples.—What is the momentum of a train weighing 71·4 tons and moving at 30 miles per hour?

71·4 tons = 160,000 pounds, and 30 miles an hour is 44 feet per second. The required momentum is therefore

$$\frac{160{,}000}{32} \times 44 = 220{,}000.$$

If the train attained its velocity in 5 minutes after starting from a station, what must have been the mean force acting on it during that time?

$ft = mv$, and t = 5 minutes = 300 seconds, hence

$$f = \frac{220{,}000}{300} = 733 \text{ pounds} = 0{\cdot}327 \text{ tons.}$$

If we take the weight given in the question as that of the train exclusive of the engine, our answer, 0·327 tons, will be the actual pull in the pull bar, or through the couplings, which the engine must have exerted during the five minutes in order to have given the train the supposed velocity in that time. Of course we should get just the same result from our original equation $f = ma$ (p. 220). For m = 5000 and $a = \frac{v}{t} = \frac{44}{300}$, so that f in tons $= \frac{5000 \times 44}{2240 \times 300} = 0{\cdot}327$ as before.

The tup of a steam hammer, weighing 4 tons, is allowed to fall 7 ft. on to a piece of iron, which it compresses. The duration of the blow, *i.e.* of the compression, is half a second. What is the average compressive force during that time?

By equation 9 of p. 180 the velocity of the weight when it strikes the iron is $v = \sqrt{2as}$; s being 7 ft. and a being here the acceleration due to gravity, or g, less frictional resistances, we may say 28 foot-seconds per second. The velocity v is therefore $\sqrt{2 \times 28 \times 7} = \sqrt{392} = 19 \cdot 8$ ft. per second. The mass of the tup is $\frac{4 \times 2240}{32} = 280$ units. Its momentum when it strikes the iron below it is therefore $280 \times 19 \cdot 8 = 5544$, which is numerically equal to the impulse $f t$. The required *mean* force f is therefore $\frac{5544}{0 \cdot 5}$, (the time being half a second,) and is therefore 11,088 pounds, or 4·95 tons.

Had the iron been cold in the example just given, so that it would not have "given" appreciably under the blow, the *time* of contact would have been much shorter, and the force or pressure of the blow very much greater, and, on the other hand, if the material had been softer and allowed the blow to last longer, its force would have been correspondingly less. The actual "force of the blow" which a steam-hammer (or any other apparatus which gives blows) can give does not therefore depend on the weight of its tup, nor is it in the least a fixed quantity for any given hammer, or pile-driver, or monkey-engine, or whatever it may be. The mass is a fixed quantity, and its velocity has some maximum value in each case, so that the momentum of the falling or striking body in any given hammer cannot exceed some determinable value. But the actual pressure which the falling tup exerts against whatever is below it,—which is what is generally meant by the force of the blow,—will depend on the duration of contact, and this will obviously depend on the hardness or compressibility of the material struck, and not in the least on the striking apparatus itself. If, therefore, in speaking of a 4-ton hammer, or 10-ton hammer, we mean a hammer in which the weight of the tup is 4 or 10 tons respectively, we are quite justi-

fied in using the expression. But if we mean that it can exert a pressure, in striking any object, of 4 or of 10 tons, or any other quantities proportional to these, we are comparing quantities which are incommensurable, and are just as much talking nonsense as if we compared the work done by two steam-engines by the weights of their fly-wheels.

The example on p. 235 would not have been essentially altered had the steam hammer been double acting, that is, had it had steam above as well as below its piston. Let us suppose that the hammer had a 20″ cylinder and that the average pressure above the piston on the down stroke was 55 lb. per square inch, and find, other things remaining as before, the average compressive force exerted during its blow.

We have still $v = \sqrt{2as}$, but a is now an acceleration due not only to gravity but also to the steam pressure. This latter is in total $55 \times \cdot785 \times 20^2 = 17270$ pounds, and the acceleration produced by it on the mass of the tup is $\frac{f}{m} = \frac{17270}{280}$ = (say) 62 foot-seconds per second, or about double the acceleration due to the weight alone. The total value of a is therefore $28 + 62$ or 90 foot-seconds per second, and $v = \sqrt{2 \times 90 \times 7} = \sqrt{1260} = 35 \cdot 5$ feet per second. The momentum of the tup would be $280 \times 35 \cdot 5 = 9940$, and the *mean* compressive force $\frac{9940}{0 \cdot 5} = 19880$ pounds or 8·87 tons.

Examples involving angular accelerations and momenta will be found at the end of the next section.

§ 32. MOMENTUM AND MOMENT OF INERTIA OF A RIGID BODY.

In the last section we have considered the momentum and moment of inertia of a body on the supposition that all its mass was concentrated in one particle having a definite radius and linear velocity. Different points in a body have, of course, in most cases, different radii and

different velocities—it is, therefore, necessary for us to find out *which* particular point in a body may be taken to represent it in this fashion, and indeed to see whether we are justified in assuming the existence of such a point at all.

The momentum of a rigid body is equal to the sum of the momenta of its different particles. If we distinguish the masses and velocities of these by suffixes, we may write for the whole momentum $m_1v_1 + m_2v_2 + m_3v_3 + \ldots$ &c. Using the sign Σ to denote the summation of all these quantities, we may therefore write shortly for the whole linear momentum Σmv.[1] In the case of a body having only a motion of translation, the value of v is the same for all its points, so that, writing M for the sum of the masses of all the particles, or Σm, we have $\Sigma mv = Mv$.

In the more general and important case, however, this is impossible. In this case the body is, moreover, rotating about an axis,[2] the directions of (linear) motion of its different points vary, and the computation of Σmv would not be very convenient, nor would the result, when obtained, be specially useful. We may, if it be wanted, substitute $v_a r$ for v (see p. 231), so that Σmv becomes Σmrv_a, and as all points have the same angular velocity v_a, we have $\Sigma mv = v_a \Sigma mr$. But with a rotating body this quantity is not of the greatest importance. Here we generally require to know not the linear but the *angular* momentum of each particle or of the whole body. For any particle we have seen in the last section (p. 231) that this was equal to $mvr = mv_a r^2$. Taking the second of these expressions (as the one more generally useful) and summing up for the

[1] To be read "Sum *mv*."

[2] The limitation is still supposed that the body has only plane motion; the more complex and for us less important conditions of screw motion will be briefly considered later on.

angular momentum of the whole body as before, we may write it $v_a \Sigma mr^2$, because v_a, the angular velocity, is the same for all the particles of the body. But Σmr^2 is the sum of the moments of inertia (see p. 232) of all the particles of the body, or the moment of inertia of the body itself about the virtual axis, so that (with this meaning for I) we may say that the angular momentum of a body is equal to $v_a I$, the product of its angular velocity into its moment of inertia about its virtual (temporary or permanent) axis.

The linear momentum of a body having a motion of translation is therefore the product of its mass and linear velocity Mv, and this quantity measures its *quantity of linear motion.* The angular momentum of a rotating body is the product of its moment of inertia about its virtual axis and its angular velocity, Iv_a, and this quantity measures its *quantity of angular motion.*

In the one case the linear velocity v, in the other the angular velocity v_a, is constant. In the one case the momentum is proportional to the mass M simply, in the other to a second moment of the mass in respect to a particular axis, or to the moment of inertia I.

In the first case, the whole mass of the body may be supposed, so far as mere quantity of motion goes, to be concentrated at any one of its points, for all those points have the same linear velocity, and a mass M (equal to the whole mass of the body) concentrated at any one of them will have a momentum Mv, equal to the momentum of the whole body.

But sometimes we require to know the momentum *in some one direction* of a rotating body. This momentum will be (see p. 237) Σmv, where v is not the whole linear velocity of each particle, but the component of that velocity in some particular direction. We may assume the body to

consist of any number of small particles, each having the same mass, so that Σm is simply M, and in order therefore to express the whole quantity Σmv in the form Mv, we may take M the whole mass, and v the mean of the velocities, resolved in one direction, of all its particles. If, therefore, we require to suppose the whole mass M concentrated at one point without change in the momentum, we cannot take that point *anywhere* in the body, but must find for its position that particular point whose velocity in any direction is the mean of the velocities of all the points of the body in that direction. This point is simply (see § 30) the centre of gravity or *mass-centre* of the body.

We may sum this matter up by saying that a body of mass M, whose mass-centre has a velocity v in any direction, has the same linear momentum in that direction as a particle of equal mass would have if it were placed at the mass-centre and had the same velocity in the given direction.

The body may have linear momentum in any direction, and the amount of this momentum can be found at once if the component of the velocity of its mass-centre in that direction be found. When a body has a motion of translation it is only the momentum in the actual direction of motion which is generally of much importance.

When a body is turning about an axis passing through its mass-centre, (that is, when the mass-centre coincides with the virtual centre, as in the case of a fly-wheel or any such body), that point, being stationary, has no velocity, and therefore the whole body has no *linear* momentum in any direction whatever. It has momentum of course, for it has motion, and its whole momentum is still represented by the quantity Σmv, but the velocities of its particles not only vary in magnitude but in direction. In this case, and indeed in most cases of rotative motion, it is most convenient to

measure the quantity of motion in angular units, *i.e.*, to determine the angular momentum $v_a \Sigma mr^2$, or $v_a I$.

To find at what point or points we may assume the whole mass of the body to be concentrated in this case, we must examine more closely the nature of the moment of inertia I. The moment of inertia of a body about any axis whatever can be found exactly by the aid of the differential calculus, or with sufficient closeness for our purposes by graphical methods. It is always, however, a more or less lengthy process to make the calculation. When, however, the moment of inertia of a body relatively to an axis passing through its own mass-centre is known, it is always, as we shall see directly, a very easy matter to find its moment of inertia about any other axis. Hence it is generally most important to know this moment for any body in respect to an axis passing through its mass-centre. At the end of this volume is given a table of the values of such moments, to save calculation, for bodies of certain simple and important forms.

The mass-centre of a body has among other properties this, that it is a point so situated in a body that its distance from any plane whatever is the average of the distances of all the other points in the body from the same plane; this indeed is one of the definitions of the point. Its distance from any plane drawn through itself is zero; hence the average distance of all the points in any body from any plane drawn through its mass-centre is also zero. Let us now take any body,[1] as Fig. 102, of which C is the mass-centre. Let c be an axis passing through the mass-centre in any direction whatever, and let o be any axis parallel to c, about which the moment of inertia of the body has to be

[1] The figure shows the body as a thin lamina merely for convenience of drawing; it may be solid, of any form whatever.

found. Further, let OC be the perpendicular distance between the two axes, M any point in the body, and MP the perpendicular distance from M to the line OC.[1] Then, as OPM is a right-angled triangle, $OM^2 = MC^2 + OC^2 + 2OC.CP$. CP is the perpendicular distance of the point M from a plane passing through c and normal to OC, and therefore if we take for M in succession all the other points in the body the mean value of the different distances, CP, so obtained must be zero. Therefore the mean value[2] of $2OC.CP$. for all the points in the body must also be zero, and we may say that taking in succession all the points of the body, and calling each in turn M, the mean value of $OM^2 = OC^2 +$ (mean value of MC^2), OC being a distance

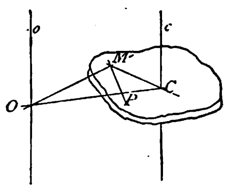


FIG. 102.

that is constant for all cases. If we now write r_o for the distance of any point M from the axis o, r for its distance from the axis c, and r_1 for the distance OC between the two axes, then the moment of inertia of the body in respect to the axis o is $\Sigma m r_o^2$, and by the equation just given, $\Sigma m r_o^2 = \Sigma m r_1^2 + \Sigma m r^2$. The last term $\Sigma m r^2$ is simply the value of

[1] The points $OMCP$ may be taken in any one plane at right angles to o and c. If this plane does not pass through C, then instead of C in the proof must be taken the point when the plane cuts c.

[2] It must be remembered that $OC.CP$ will be intrinsically positive or negative according to whether the sense of OC is the same as or opposite to that of CP. In the figure the senses are opposite, so that the product $OC.CP$ is negative.

the moment of inertia of the body about the axis from which r is measured for each point, *i.e.*, about an axis (such as c) passing through its mass-centre. In the second term ($\Sigma m r_1^2$), r_1 is the same ($= OC$) for all points in the body, so that $\Sigma m r_1^2$ is simply equal to $M r_1^2$, where M is the whole mass of the body. It is therefore simply equal to the moment of inertia about the given axis o which the whole body would have if its mass were all concentrated at its mass-centre C. The result of the whole matter is therefore that **the moment of inertia of a body about any axis whatever is equal to its moment of inertia about a parallel axis through its mass-centre, plus the moment of inertia of the whole mass, considered as concentrated at its own centre, about the same axis.**

An example may probably make this matter more clear. Let it be required to find the moment of inertia of a solid cast-iron cylinder 8 inches diameter and 12 inches long, about an axis parallel to its own and 2 feet distant from it. The moment of inertia of the cylinder about its own axis (which of course passes through its mass-centre) is 0·28, taking pounds and feet as units, and assuming the iron to weigh $\frac{1}{4}$lb. per cubic inch. Its weight is 150 lb., its mass therefore $\frac{150}{32} = 4{\cdot}7$, and the moment of this mass, concentrated at the mass-centre, about the given axis, is $4{\cdot}7 \times 2^2 = 18.8$. The actual moment of inertia of the cylinder about the new axis is therefore $0{\cdot}28 + 18{\cdot}8$, or 19·08. In this case the error in assuming the whole mass to be simply concentrated at its mass-centre and neglecting the term $\Sigma m r^2$ of the equation on page 241, is about $1\frac{1}{2}$ per cent.

If the moment of inertia of a body about any axis be

divided by its mass, we get a quantity whose square root is equal to the radius at which the whole mass would have to be concentrated in order to have as a particle the same value of I which it actually has as a body. In the example just given, $\frac{19{\cdot}08}{4{\cdot}7} = 4{\cdot}06$ and $\sqrt{4{\cdot}06} = 2{\cdot}015$ feet or $24{\cdot}18$ inches. If, therefore, for the cylinder with its axial radius of 24 inches, we substituted a particle of equal mass ($4{\cdot}7$) at radius $24{\cdot}18$ inches, the moment of inertia would remain unchanged. Such a particle would therefore have the same quantity of angular motion, or angular momentum, as the actual body, and might replace it in any problems in which angular momentum occurs.

In symbols this may be written $\frac{\Sigma mr^2}{\Sigma m} = k^2$, where k is the radius just mentioned. This radius is commonly called the "radius of gyration" or "radius of oscillation" of the body. Such an expression is rather awkward, and unrelated to any of the properties of k with which we are specially concerned; we shall therefore call it the **radius of inertia** of the body, under which name it has already been referred to in advance in § 23.

It will be noticed that it was not any *one particular* particle which was supposed (above) to take the place of the whole body, but any particle whatever having the required radius k. This is a radical point of difference from the condition of the mass-centre, which is one point only in the whole body. Under certain circumstances it may be convenient to take, among all possible points of radius k, that particular one which lies on the line joining the mass-centre to the virtual centre, *i.e.*, upon the virtual radius of the mass-centre. Such a point may be called the **centre of inertia** of the body. It must be noticed, however,

that its use is only a matter of convenience, there being nothing special about the properties of this point as compared with those of other points at the same radius. In the case of a body (as a fly-wheel) rotating about an axis through its mass-centre, the mass-centre has no virtual radius, so that no point exists which can be called specially the centre of inertia under the definition just given.

It is also to be noticed that when a body, as is the general case, is turning at successive instants about different virtual centres, its radius of inertia is continually changing also. For each changed position of the body, therefore, a fresh set of points lie at the radius k, and the centre of inertia (where it exists) not only has a new virtual radius, new direction of motion and new velocity, but *is itself a different point on the body.* It is not possible therefore to picture the path of the centre of inertia as we can the path of the mass-centre, or the point-path (p. 35) of any other point in the body.

We may now work out some further examples to show the use of the quantity I, and the handling of questions as to angular accelerations and momenta.

The fly-wheel of a stationary engine attains a velocity of 70 revolutions per minute in 50 seconds after starting. Its moment of inertia about its axis is 13,000 (the units being feet and pounds). What must have been the average force at the radius of the crank pin, say 18 inches, to cause this change of angular momentum?

Here the value of mr^2 is given in the question as 13000, and v_a, the angular velocity, will be $70 \times 2\pi \div 60$, or $7 \cdot 33$ angular units per second. Hence $f r t = v_a I = 7 \cdot 33 \times 13000 = 95290$, and dividing by the time t, or 50 seconds, $fr = \dfrac{95290}{50} = 1906$ foot-pounds, and r being $18''$, $f = \dfrac{1906}{1 \cdot 5} = 1270$ pounds. This is the average force at the radius of the crank pin. We shall see presently that the average piston pressure must exceed this in the ratio $\dfrac{\pi}{2} : 1$, so that if we had to find

the pressure per square inch of piston which must have been taken up simply in changing the momentum of (that is in accelerating) the fly-wheel, when the engine was starting, we could readily find it. In the case supposed the diameter of the cylinder might have been about 18 inches, the area for which is 254 square inches. The required mean pressure on this area would therefore be $\frac{1270 \times \frac{\pi}{2}}{254} = 7 \cdot 8$ pounds per square inch. This pressure would be required solely for the acceleration of the fly-wheel, and would have to be deducted from the total pressure on the piston, as long as the fly-wheel was receiving momentum, in order to find what pressure was available during that time for driving the rest of the engine as well as the machinery of the work-shop, or whatever the engine had to drive.

In the same case, what brake pressure applied at the periphery of the wheel, say 5 feet radius, would bring it to rest in half a second?

Here we have the change of angular momentum the same as in the last case, but a decrease of 95290 instead of an increase. The pressure required will be $\frac{95290}{5} \div \frac{1}{2} = 38120$ (about). This forms a good illustration of the way in which the pressure required varies inversely as the time, mentioned on p. 235.

A solid disc of cast-iron is 20 inches in diameter and 2 inches thick: it revolves in its own plane about an axis through its own centre of gravity at the rate of 100 revolutions per minute. What force applied at its periphery can double its velocity in 2 seconds?

The disc weighs 160 pounds, its mass is therefore 5. Its moment of inertia is $1 \cdot 73$ (feet and pounds being the units), and its angular velocity is $10 \cdot 5$. Its gain of angular velocity is therefore also $10 \cdot 5$, and of angular momentum $10 \cdot 5 \times 1 \cdot 73$ or $18 \cdot 2$. The force required is therefore $f = \frac{v_a I}{rt} = 18 \cdot 2 \div \frac{10}{12} \div 2 = 10 \cdot 9$ pounds.

A connecting rod weighs 700 pounds, so that its mass is 22. Its moment of inertia in its plane of motion about an axis through its mass-centre is found by calculation to be 63. What is its angular momentum and what its radius of inertia? (Fig. 89 may be referred to to illustrate this question, but the figures given here are taken from another example.)

The mass-centre of the rod being known, its virtual radius must first be measured. This is found to be $3 \cdot 55$ feet. The value of I is therefore $63 + (22 \times 3 \cdot 55^2) = 340$. The angular velocity of the connecting-

rod is found by finding the angular velocity of one point in it. Most conveniently this point is the centre of the crank pin. Suppose the stroke of the engine to be 2 feet and the speed of rotation of the shaft 75 revolutions per minute. Then the linear velocity of the crank pin is $\frac{2 \times \pi \times 75}{60} = 7{\cdot}85$ feet per second. Its angular velocity as a point in the connecting-rod (and therefore the angular velocity of all other points in the connecting-rod) is $7{\cdot}85 \div$ its virtual radius as a point in the connecting-rod. This radius is found by measurement to be $5{\cdot}3$ feet.

The angular velocity of the connecting-rod is therefore $\frac{7{\cdot}85}{5{\cdot}3} = 1{\cdot}5$ nearly. The angular momentum of the connecting-rod is $1{\cdot}5 \times 340 = 510$. The radius of inertia is $\sqrt{\frac{I}{m}} = \sqrt{\frac{340}{22}} = 3{\cdot}95$ feet. The radius of inertia is thus $0{\cdot}4$ feet greater than the virtual radius of the centre of gravity of the rod.

§ 33. WORK AND ENERGY. RATE OF DOING WORK. HORSE POWER.

We have seen that to alter the quantity of motion possessed by a body, *i.e.*, to give it any acceleration, requires not only the expenditure of force, but the expenditure of force over some finite interval of time. The product of force and time, $f\,t$, we have called *impulse*, and have seen that there was equality between any impulse and the change of momentum produced by it, ($f\,t = m\,v$, — see § 31). But during the finite time over which the force has acted, the body must have moved through some space or distance. This distance can be expressed in terms of the time, for the velocity with which it is passed over is known. We get in this manner some new relations which are of still greater practical importance in connection with machines than the

relations of impulse and momentum. Suppose, for instance, that a body of mass m is accelerated from v_1 to v_2 by a force f acting for t seconds, we know that $f t = m\ (v_2 - v_1)$. But the space passed through during the operation is equal to the mean velocity of the body, multiplied by the time taken up by the change, or $s = \frac{(v_2 + v_1)}{2} t$. By substitution we have therefore, $f s = m \frac{(v_2^2 - v_1^2)}{2} = \frac{w\ (v_2^2 - v_1^2)}{2g}$. If the body started from rest, so that $v_1 = 0$, then (writing v for the final velocity v_2) $f s = \frac{m\, v^2}{2} = \frac{w\, v^2}{2g}$.

The quantity $f s$, the product of a force (pounds, tons, etc.) into a distance (feet, inches, etc.) through which it acts, is called the **work** done on the body, and is measured in compound units called foot-pounds, inch-tons, etc., according to the standards employed for force and distance. This measurement of work in foot-pounds is to be distinguished from our former measurement of static moment in similarly named units. A static moment is the product of a force[1] into a distance *normal* to its direction; work is regarded as the product of a force into a distance along its own direction, *i.e.*, the distance through which it is exerted on the body moved by it. There is practically no chance of confusing the two kinds of foot-pounds, so that no practical drawback occurs from the double use of the name.

A simple case of work is that of a falling weight, where s is simply the distance fallen through by the weight, and v is the velocity it has attained starting from rest, under the action of gravity. A train starting from a station is another simple case. If we here take f as the pull on the drawbar connecting the engine with the first wagon, and s the distance run

[1] Or other "directed" quantity.

at the time any velocity v is attained, then fs is the work done by the engine in accelerating the train, independently of the work which the engine has to do in causing its own acceleration, and in overcoming certain frictional resistances to be presently considered.

In § 31 we examined the equality $ft = mv$, one side of which we called the impulse, and the other the momentum. We have a somewhat similar case before us now, $fs = \frac{mv^2}{2} = \frac{w\,v^2}{2g}$.[1] The left hand quantity we call the *work done on the body.* The right-hand quantity, which is numerically equal to it, we call the *energy received by the body in virtue of that work.* It is not necessarily the whole energy possessed by the body, but in the general case is simply the additional energy received by it in virtue of the acceleration from v_1 to v_2. The added energy may be small or great in proportion to the original energy, according as the added velocity is small or great in proportion to the original velocity.

If we wish to bring the body back to its original condition of rest, or of velocity v_1, we must take away from it this quantity of additional energy $\frac{w\,v^2}{2g}$ or $\frac{w\,(v_2{}^2 - v_1{}^2)}{2g}$. Or if conversely this quantity of energy be in any fashion taken away from the body, it will return to its original condition of rest or of velocity v_1. Hence, we commonly, and justifiably, speak of this as work or energy *stored up in the body*, and of the taking away of this energy as its *restoring* to other objects.

Using these terms we may therefore say that **to make**

[1] The latter form will be generally used instead of $\frac{mv^2}{2}$ because in practical problems the weight, and not the mass, is always one of the data.

any addition v to the velocity of a body of mass m, or weight w, we must do upon it an amount of work ($f\,s$) equal to $\frac{m\,v^2}{2}$ or $\frac{w\,v^2}{2g}$, and that this amount of work will remain stored up in the body as long as it retains its added velocity, but must be restored by it to other objects before it can regain its original velocity. Conversely to diminish the velocity of a body we must take away from it a portion of the energy which it possesses in virtue of its velocity, and must again restore this to the body if it is again to move as fast as before.

Just as we speak of doing work on the body while it is being accelerated, so we say that the body itself does work —on something else—during the process of restoring energy. Thus we say that a body whose velocity is v, no matter when or how it has attained that velocity, has stored up in it in virtue of that velocity a quantity of energy $\frac{w\,v^2}{2g}$, and that this is exactly equal to the quantity of work which the body could do for us, suitable appliances being provided, before it was brought to rest. This quantity is often called the **kinetic energy** of the body.

If a body be in a position from which it can be allowed to fall, and at the end of the fall to do work for us, it is sometimes said to possess *energy of position*, and this energy is measured by the energy which it possesses at the bottom of its fall. An ordinary pile-driver with a monkey weight is an illustration of this case. The weight at the top of its lift is said to possess "energy of position" equal to the product of the lift and the weight. This expression seems to be misleading and (at least in such cases as we have to

do with) valueless. The only measure of energy of position which would seem reasonable, is the whole energy which a body would possess in virtue of falling from its present position to the centre of the earth. In ordinary mechanical cases we do not require to look at things this way. The monkey weight possesses energy when it hits the pile-head corresponding exactly to its velocity at that instant, and therefore in no way forms a different case from those we have been considering.

A body cannot have its velocity increased without having work done on it, and stored up in it; it cannot have its velocity diminished without re-storing some of that work, and thereby diminishing its own kinetic energy.

But on the other hand, work can be done in other ways than in causing the acceleration of masses. We see not unfrequently force occurring without apparent acceleration. It does not require long examination to see that this is no real contradiction to the statement (p. 217) that we knew force only as the cause of acceleration. A body is being moved at a uniform velocity along a flat surface. The friction against the surface (§ 71) resists its motion, and if it were left to itself it would rapidly come to rest. The force acting on it is necessary to prevent this negative acceleration, and must be therefore of such magnitude as to cause an equal positive acceleration, although the body is actually moving without any acceleration of velocity at all. In reality, the body is moving under the action of two equal and opposite forces, causing equal and opposite accelerations, which together may cause either rest or (if one force be allowed for one instant to preponderate by any small amount) motion with uniform velocity, and therefore without acceleration.

In such a case—and this is a most frequently occurring

case in practice—no work is done *on the body* itself by the force, and no energy is stored up in it.

Following Rankine's nomenclature, we may call any two such forces acting on a body **effort** and **resistance.** They are opposite and (if the body has uniform velocity) equal. The effort is always the driving force, or force acting in the direction in which the body is moving. In these terms we may say that, in the case supposed, the work done by the effort is equal to the work taken up by the resistance. No work is left to be expended on, or taken up by, the driving body itself. It would be always possible, even in these cases, to calculate the work expended in terms, not of the velocity of the body, but of the rate at which it would have lost that velocity had the effort not been acting. There is no object, however, in doing this, for the work done is simply the product of the effort into the distance through which it has acted, and is thus given at once in proper work units.

Thus the work done in a minute by a locomotive in drawing a train weighing 120 tons at a uniform rate of thirty miles an hour on a level line, the resistance being 12 lbs. per ton, would be

$$120 \times 12 \times 2640 = 3{,}801{,}600 \text{ foot-pounds.}$$

Similarly the work done in an hour at the crank pin of an engine whose radius is 18 inches, and which revolves at a uniform rate of 50 revolutions per minute against a uniform resistance of 7500 pounds, would be

$$7500 \times 50 \times 60 \times 3 \times \pi = 212{,}000{,}000 \text{ foot-pounds}$$

(about).

Very frequently work is done simultaneously against a

direct resistance and a resistance due to acceleration. There is, of course, no difficulty in calculating the work done in any such case, by finding the two quantities separately and adding them together. Thus let it be required to find the work done by a winding engine in lifting a cage weighing two tons through 150 feet, during which time the cage (originally stationary) has had its velocity increased to 10 feet per second. The mere lifting of the cage must have taken up $2 \times 150 = 300$ foot-tons or 672,000 foot-pounds of work, quite independently of the acceleration. In addition to this the work now stored up in the moving cage, which must all have been supplied by the engine, is

$$\frac{2 \times 2240 \times 10^2}{2 \times 32.2} = 6950 \text{ foot-pounds.}$$

The total quantity of work is therefore

$$672{,}000 + 6950 = 678{,}950 \text{ foot-pounds.}$$

This question affords a good illustration of a very important point connected with the expenditure of work in a machine. The body accelerated—here the cage as it is lifted—has eventually to come to rest. In order to do so it must get rid of all the work stored up in it in virtue of its velocity. This case occurs in all machines, where accelerated bodies always eventually come to rest or resume their original velocity. The kinetic energy, when the negative acceleration comes, may be got rid of in two ways. It may be expended in doing work on the body in which it is stored up, in which case the body simply continues moving until all the stored-up energy is exhausted, and then stops. Or some fresh work may be provided for it to do, upon which it expends itself until it is exhausted as before.

In the case of the cage just supposed, if the whole 6950

foot-pounds stored up were to be expended in continuing to lift it, (the effort of the engine suddenly ceasing,) it would only rise

$$\frac{6950}{2 \times 2240} = 1{\cdot}55 \text{ feet}$$

before the whole kinetic energy would be exhausted, and the carriage would stop and then begin to fall unless supported by the engine or otherwise.[1]

In the case of the train on p. 251, if the connection with the engine suddenly ceased, the train would move on a very great distance. For the energy stored up in it (30 miles an hour = 44 feet per second)

$$\frac{120 \times 2240 \times 44^2}{2 \times 32{\cdot}2} = 8{,}131{,}000 \text{ foot-pounds,}$$

and this would suffice to carry the train against the small resistance of $120 \times 12 = 1440$ pounds, through a distance of 5650 feet, or nearly a mile, before it would come to rest. This would be out of the question practically, and therefore an artificial resistance to the motion of the train is provided by the brakes. As the magnitude of such a resistance can be made independent of the velocity or (within certain limits) of the mass of the train, the train can be stopped within any desired distance by making the brake pressures sufficiently great.

We have examined in this section the methods of measuring quantities of work. It is clear that any quantity of work, however great, can be done by any force, however small, if it only act over a sufficiently great distance. The

[1] In such a case in practice most of the energy stored up by a winding engine is stored up in the rotating mass of a large fly-wheel, not in the rising carriage.

mere measurement of work in foot-pounds would therefore not afford us any means of comparing the apparatus by which the work was done. For this purpose we compare, not the absolute quantities of work done, but the *rate at which that work has been done*, that is, the number of foot-pounds of work done in a unit of time. The unit of time employed here is almost invariably the minute, and for many purposes we may measure the rate at which work has been done simply in foot-pounds per minute. But for many engineering problems this would give quantities inconveniently large, and therefore in engines, by common consent, we measure the rate at which work is done in units called *horse-power*, one horse-power being taken as equal to 33,000 foot-pounds of work done in one minute. It is essential to remember that a horse-power is not a unit of work or a quantity of work, but a quantity of work done in a certain time. It measures, not work, but the rate at which work is done, in exactly the same way as acceleration measures, not velocity, but the rate at which velocity is gained.

We may use to illustrate this some of the problems already worked out in this section. The locomotive mentioned on p. 251, does 3,801,600 foot-pounds of work *in a minute* by hypothesis. The horse-power it is exerting is therefore got simply by dividing that figure by 33,000, and is 115 (about). The horse-power of the engine in the next example is $\frac{212,000,000}{33,000 \times 60} = 107$ (nearly).

In the case of the winding engine the determination of the horse-power involves the finding of the *duration* of the operation described, which has not been given. We have given that the velocity of the cage has been increased from 0 to 10 feet per second while it has travelled 150 feet. The duration is therefore to be obtained from the equation

$t = \frac{2s}{v} = \frac{300}{10} = 30$ seconds. Work has therefore been done at the rate of 678,950 foot-pounds in half a minute, which is equivalent to 41·1 horse-power.

In the problems of this section only work against linear acceleration has been considered. The consideration of angular acceleration does not require any fresh treatment. Such problems as the following occur, and may be solved at once without further explanation.

How much work is stored up in the fly-wheel of an engine making 65 revolutions per minute if the wheel has a radius of inertia[1] of 7 feet 6 inches and a weight of 5 tons? Here v is to be taken as the linear velocity of the fly-wheel rim at the given radius, and is 51 feet per second, the whole kinetic energy of the wheel being

$$\frac{5 \times 2240 \times 51^2}{2 \times 32{\cdot}2} = 452{,}300 \text{ foot-pounds (about).}$$

If the engine attained this velocity in 90 seconds after starting from rest, what horse-power must have been taken up merely in accelerating the fly-wheel? This is nothing more than $\frac{452{,}300}{33000 \times 1{\cdot}5} = 9{\cdot}10$ horse-power (about).

We shall only remind the student, in conclusion, that energy appears in many different, but transformable, forms. By changing "mechanical energy" into heat, for example, the motion of a body as a whole becomes the motion of its vibrating atoms. This matter does not, however, form a part of our present subject.

[1] In such a case as this the radius of inertia is often taken as the mean radius of the wheel rim, in which the greater part of the mass is concentrated. See § 32.

§ 34.—SUMMARY OF CONDITIONS OF MOTION POSSIBLE IN A MECHANISM.

What has been said about velocity, acceleration, etc., in the last few sections does not apply specially to mechanisms, but is quite general in its bearing. In what follows, however, we shall limit ourselves to such applications only of the general principles now established as come within our proper subject. We have already seen that in machinery every motion is *constrained* (§ 1); we have examined at some length the meaning of this condition, and have already made no little use of it in considering the motions for their own sake. We shall now find it no less useful and convenient to us in dealing with dynamic than formerly in dealing with kinematic problems.

The bodies with which we have to deal at present are at any instant either (i) stationary, (ii) undergoing simple translation, or (iii) undergoing simple rotation about a virtual axis, and we know that condition (ii) is only the special case of (iii) where the virtual axis is at infinity. The position of the virtual axis, or the direction of translation, is in every case determined absolutely by the form of the connection between the bodies or links constituting the mechanism. Disregarding forces which can destroy the mechanism by distorting it—the consideration of which falls under what is usually called the science of the Strength of Materials — we have, therefore, said (p. 5) that the direction in which any point or body in a mechanism is moving at any one instant is independent of the forces acting on that point or body. Only the *magnitude* of the velocity or acceleration depends on the forces, while

such a magnitude if fixed for one point is fixed for all the others.

Besides the case of stationary bodies, such as are in mechanisms, of course, the fixed links, we have then only two cases to deal with, bodies whose motion is a simple translation, and those whose motion is a simple rotation about a fixed axis at a finite distance.

In the former case the body has no angular velocity or acceleration (p. 214) but it may be either moving with a constant linear velocity or undergoing linear (tangential) acceleration, and if it is undergoing acceleration, the acceleration may be either constant or varying.

In the latter case the body must have angular velocity, and may or may not have angular acceleration, either constant or varying. Every particle in it must always have radial acceleration, but in one special case (where the virtual centre coincides with the mass-centre) the sum of the radial acceleration of all the particles is zero, so that the body as a whole has no such acceleration (p. 215). If the angular velocity of the body be constant,[1] the linear velocity of every point in it must be constant, and its radial acceleration must be constant. If it has, on the other hand, angular acceleration, every particle in it must also have linear acceleration, and its radial acceleration instead of being constant will be varying. The linear acceleration of each point will be constant or varying according as the angular acceleration of the body be the one or the other.

If the linear or the angular velocity of a body is undergoing increase, that is, if the body is undergoing positive tangential or angular acceleration, some equivalent expenditure of force is continually taking place, and work is being

[1] Assuming it to be turning about a permanent centre.

expended on the body and stored up in it. If, on the other hand, the acceleration be *negative*, the speed decreasing, the body must be continually parting with or re-storing energy. In the former case the driving effort exceeds the resistance, in the latter the resistance exceeds the effort.

If, however, the body be moving with uniform velocity, the effort and resistance are equal. Work is being done equivalent to the distance through which the effort is exerted, but this work is not stored up in the body, and at the end of such an operation the moving body contains only the same kinetic energy as at the beginning. The work done by the effort may have been done in lifting a weight (here the resistance), or it may have been expended in cutting or rubbing some material, or it may have been wholly or (as happens in almost every case) partially converted into heat, or converted into electricity. In whatever way it has been taken up it has passed away from the particular machine or mechanism on which the effort acted, and to that extent has become irrecoverable.

CHAPTER VIII.

STATIC EQUILIBRIUM.

§ 35.—CLASSIFICATION OF THE FORCES ACTING ON A MECHANISM.

ON every link in a mechanism, including, of course, the fixed link, there are usually a number of forces acting. We shall find it possible to classify the possible force conditions as simply as we have classified the possible conditions as to velocity and acceleration.

In the first place, the whole forces acting upon and between the links of any mechanism can be divided into two classes, between which it is quite easy to distinguish. Some of the forces, namely, are entirely external to the mechanism itself, and both in direction and magnitude may be independent of it; such forces are called **external forces.** The weight hanging from a crane, the resistance of a piece of iron to the edge of a cutting tool, the pressure of steam on a piston, are examples of such external forces. The weights of the individual links in any mechanism also fall into this category.

When any such external forces act at different points upon a mechanism—whether or not they cause the mechanism to move—they give rise to other forces acting from link to link of the mechanism, determined in magnitude by

the external forces, but fixed in direction solely by the nature of the mechanism itself. These forces, in default of a better name, we shall call **pressures.** It would be misleading to call them "internal" as opposed to the "external" forces, for although they are internal in respect to the mechanism as a whole they are external to its links individually. In such an example as that of Fig. 103 the *external forces* are the two weights W_b and W_d, whose magnitudes and directions may be anything whatever[1] that we choose. Besides these there are in the mechanism forces

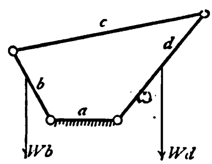


FIG. 103.

exerted by each link upon those next adjacent to it, whose total magnitudes are determined by the external forces, but whose directions and relative magnitudes are fixed by the mechanism itself. These are the *pressures*. The pressures exerted by *b* and *d* upon *c* are forces external to *c*, although not external to the mechanism. Similarly the pressures exerted by *a* and *c* upon *b* are external to it just as much as the external force W_b. But again they are not external to the mechanism, and therefore do not receive the name of external forces.

Pressures being by definition actions between adjacent links, occur always at the surfaces or lines of contact of the pairs of elements. We might say that they occurred always

[1] "Anything whatever," because we make no pre-supposition that the mechanism shall be "balanced" under them.

at the *joints* if we had only to do with turning pairs, but sliding surfaces are not generally included under the head of joints, although they are equally important to us as pairs. But it is clear that the pressures acting at different points, perhaps points very far apart, on the links, must be transmitted from point to point through the material of the links themselves. These transmitting, molecular forces might correctly be called internal forces; they have, however, received the more convenient name of **stresses,** which we shall always use to designate them.

In rigid bodies, stress may be defined as **resistance to alteration of form**; in fluids—which occasionally form part of machines, (see p. 3)—as **resistance to alteration of volume.** It is this capacity of the material of the links to exert stress in such fashion as to preserve their forms sensibly unaltered that justifies us in treating the virtual centre as a fixed point (p. 44). Could the shape of the links alter sensibly, the position of the virtual centre would be to a corresponding extent variable, the machine would become useless for its own proper purposes, and our method of examination would become inapplicable.

So long as the links of a mechanism are either stationary or moving with constant velocity, there come into question only these three—external forces, pressures, and stresses. Pressures act on each link from the next one at every pairing. The pressure of the link *a* on the link *b* is external to *b*, the pressure of *b* upon *a* at the same place is external to *a*, and so on. But no pressures can exist unless in the first instance they are called into existence by external forces acting on the mechanism. For the pressures may be taken to represent the resistance of the links (consequent on the manner in which they are connected together) to change of relative position, just as the stresses represent the resistance

of the molecules (consequent on the manner in which *they* are connected) to change of relative position. The difference between the two cases is that the links are allowed to change their positions to a very large extent, and the molecules only to a very small one. The external forces may act on only one link of the whole mechanism, or on all, or on any number of the links. The mere weight of the links themselves may form a most important part of these forces, or may (as in a horizontal steam engine) be fairly negligible in comparison with the rest. The stresses, as representing entirely intermolecular action, may be left out of account here, it being presupposed only that the links are made of such material and dimensions as will keep the stresses in them so small that their change of form under pressure may be safely neglected. The stresses will then stand in the same general relation to the pressures that the pressures do to the external forces, except that wherever an external force acts on a link along with pressures it takes exactly the position of a pressure in causing stress.

The last paragraph has contained a general statement of the relations between stress, pressure, and external force in the case of bodies stationary or moving with constant velocity. When a body has acceleration, a force not falling properly under either of these three heads has to be taken into account. A body offers no resistance to continuance of motion in its own direction with its own velocity, but it cannot be accelerated without the action of force. This fact, which is Newton's "first law," is at the foundation of our whole study of dynamics. But it involves directly the converse fact that every body simply in virtue of its existence offers resistance to acceleration. This resistance is exactly measured by the force necessary to cause the acceleration, is equal and opposite to it, stands to it, in

fact, in the relation of reaction to action. Neither can exist without the other; either may be looked at alone, but only if we do not forget that it is only half of a duality.[1]

When, therefore, any link of a mechanism is undergoing acceleration, its **resistance to acceleration**—a quantity proportional directly to its mass, as well as to the acceleration, but for which, unfortunately, we have no single word—is a force which has to be taken into account along with the rest, and which falls neither into the class of external forces nor into that of pressures, as we have defined them. We shall find presently that problems involving "resistance due to acceleration" are not more difficult to deal with than any others.

§ 36.—EQUILIBRIUM—STATIC AND KINETIC.

So long as the form of a body is not actually undergoing change—lengthening, shortening, distorting, etc.—the body is said to be in **equilibrium**. This equilibrium is called **static** if the body is stationary or moving with uniform velocity, and **kinetic** if it is undergoing acceleration.

For a body to be in static equilibrium it is necessary simply that the external forces acting upon it should not be such as could, in their united action, cause acceleration. Now the united or total action of any system of forces on a body is in every respect, except as to the stresses caused by the forces, the same as the action of the resultant or sum of that system of forces. The sum of any number of forces

[1] We talk similarly of the pressure of a girder on its abutment or of the reaction of the abutment against the girder. Neither can exist without the other, but without losing sight of the duality we often for simplicity's sake speak of only one.

may be either (i) zero, (ii) a single finite force of definite direction and position, or (iii) a couple, which has sense and has also magnitude measured as a moment, but has neither magnitude as a force nor any position or direction. So long as the forces are all in one plane, the condition always presupposed in this part of our work, no other condition than one of these three is possible.

If the sum of all the external forces be zero the body must be in static equilibrium, for zero force must cause zero acceleration.

If the sum of all the external forces acting on any link of a mechanism be a single force the equilibrium of the link is static or kinetic according to the position of that force. If the force passes through the virtual centre it can give the body no acceleration, because that point is a fixed one; no force whatever by acting on it can either make the body move or change its motion if it is already moving. In every other case a single force can and must cause the body to be accelerated. This may be summed up by saying that **if the sum of the external forces acting on any link of a mechanism be a single force, the link will be in static equilibrium only if that force act through its virtual centre relatively to the fixed link.**

If the sum of all the forces acting on any link of a mechanism be a couple, the condition of the link depends on the position of its virtual centre. If the link has a motion of translation only it will be in equilibrium, because its virtual centre is at infinity; in all other cases it must be undergoing acceleration. For looking at a couple merely as two equal, parallel, and opposite forces, there is no difficulty in seeing that it cannot cause acceleration in a body whose only possible motion is one of translation in one

particular direction. For the two forces which together constitute the couple being parallel and opposed to each other, as well as equal, have no tendency to shift the body as a whole in the only way in which (by virtue of its connection with the rest of the mechanism) it can be shifted. Whatever motion either one of them could give it is directly counteracted by the opposing action of the other. A simpler proof of this is derived from the modern treatment of a couple as an infinitely small force at an infinitely great distance, acting along "the line at infinity." Such a force must (by definition) pass through all points at infinity, and therefore through that particular point which is the virtual centre for the motion of the link. The case therefore falls within that of the last paragraph—such a force can give the body no acceleration.

If, on the other hand, the virtual centre of the link be at a finite distance only, the link cannot be in static equilibrium, for the couple can cause angular motion, and therefore acceleration, although it cannot cause simple translation. This may be proved in various ways. In the first place, the last method of the last paragraph shows us that the couple is a force *not* now acting through the virtual centre, and therefore capable of causing acceleration. Or, secondly (without making use of the infinite elements), we know that a couple may be shifted anywhere in its plane of action without altering it, *i.e.*, that it has no special position. Therefore we may suppose it so shifted that one of the two forces of which it consists passes through the virtual centre. This force can therefore give the body no acceleration, but it leaves it under the sole action of the second force, which by hypothesis can *not* pass through the virtual centre. So far as acceleration goes, the body is therefore just in the position of one acted upon by forces whose sum is a single

force. The two last paragraphs may be summed up as follows:—**If the sum of the external forces acting on any link of a mechanism be a couple, the link will be in static equilibrium only if its constrained motion be one of translation.**

When a body is in static equilibrium the external forces acting on it are generally said to be **balanced.** This expression may mean either of two different things. If the static equilibrium results from the sum of the forces being zero, it means that they are balanced among each other—that as a whole they have no tendency to move the body, because as a whole they have no magnitude. This condition occurs in structures constantly, but only extremely seldom in machines. If the equilibrium results from the sum of the forces passing through the virtual centre, the forces are *not* balanced among themselves—(for their sum is a finite force)—but are balanced by the pressures between the links of the mechanism which keep the virtual centre stationary.

When, however, the body is in kinetic equilibrium, the force (or couple) causing acceleration is often said to be **unbalanced,** and indeed the condition is not always recognised as one of equilibrium at all. But the force here, which is the sum of all the external forces in action, is balanced just as completely as the force passing through the virtual centre in the case just dealt with. Neither is balanced by what may be called **independent** as distinguished from **derived** external forces, and therefore in one sense both might be called unbalanced. But one we have just seen to be balanced by the pressures acting between the links, called into action inevitably, each in its particular direction, as soon as the external forces begin to act (p. 261), and the other (with which we are now dealing) is no less

really balanced by the resistance to acceleration (p. 263). What is generally called an unbalanced force, or couple, is therefore one which is not balanced either by external force or by pressure, but solely by the resistance of the body or bodies upon which it acts to acceleration. There is no particular harm in the use of the word unbalanced in this case, if this limitation of its meaning be kept in mind.

In speaking first of force (p. 217) we said that we measured and compared the magnitudes of forces only by the accelerations they produced or could produce. We find now that a body may be acted on by forces to any extent and yet be undergoing no acceleration. There is here, however, nothing contradictory. The conclusion which we have to draw when we see forces acting on a body which stands still or moves with constant velocity, is that the accelerations produced by those forces must be such as exactly to counteract each other, so that as regards acceleration the state of the body is the same as if no force were acting at all. It is because we know that in such a case the body receives no acceleration that we say that the forces acting on it must be such as to produce accelerations whose sum is zero, and then infer that the sum of the forces must be zero also, because forces are proportional to the accelerations which they produce on the same body. Or otherwise, as we have already put it (p. 250), if a body is visibly moving with uniform velocity, but is visibly also acted on by forces, we can only conclude that each force is producing its own acceleration, but that the forces are such that their whole accelerations exactly cancel each other. We should infer that if in such a case we took away any one of the forces, the body would at once begin to move faster or slower at a rate exactly corresponding to the now unbalanced part of the acceleration naturally due to all the remaining forces.

§ 37.—STATIC EQUILIBRIUM — GENERAL PROPOSITIONS.

The ordinary static problems connected with machinery are of two kinds: (i) the determination of the forces or moments acting on a body on the assumption that they must be such as to bring it into static equilibrium; and (ii) the finding of whether or not a body is in static equilibrium under the action of certain given forces. The first class of problems is far the most important, and we shall confine ourselves chiefly to it; the second class will not afterwards present any difficulty.

In practice the problem very generally takes this form:—Given that a mechanism is in static equilibrium, and that a certain external force or set of forces is acting upon it, to find what force in a given direction, and acting on a given link, is required to balance the given force or forces. The problem is simplest when all the forces are acting on the same link of the chain, but in practical work it often happens that given forces act on several links, and the force to be found acts on quite another. This is of course a more complex question, but we shall find that it can very readily be solved by an extension of the same methods which we shall use in the simpler cases.

The general conditions for the static equilibrium of a body having plane motion have been stated in § 36. They may now be put into more extended form in view of the problems just mentioned, and this is done in the following propositions:—

(i) **The sum of the external forces[1] may be zero.** This occurs very seldom, as it involves the very special case

[1] For definition of "external forces" see p. 259.

that the total action of all the given forces upon the link on which the unknown force acts should be in direction and position exactly opposite to that force. The case need not be separately considered, as it is most conveniently treated under one of the following :—

(ii) **The sum of all the external forces must pass through the virtual centre.** This is simply an inversion of the proposition on p. 264, and is the form of the proposition which will be found most generally useful.

(iii) **The moment** (p. 232) **of the sum of the external forces about the virtual centre must be zero,** or, what is the same thing, **the sum of the moment of all the external forces about the virtual centre must be zero.** This follows at once from (ii), for if the sum of the external forces is itself a force passing through the virtual centre, it cannot have any moment about that point. This form of the proposition is specially useful in enabling us to deal with couples, but is convenient in many other cases. If the sum of the external forces is zero, their moment (that is, the moment of their sum) is zero about every point in the plane, and not only about the virtual centre. But if the sum has any finite value it must have some moment about all points except those lying in its own direction line, and of these points the virtual centre must always be one.

It follows from (iii) that the effect of a force upon a body whose motion is one of rotation about a point at a finite distance, is not simply proportional to the magnitude of the force, but to its **moment**—the product of its magnitude and radius, or perpendicular distance from the virtual centre. A given force will produce the same linear acceleration in a body of given mass, whether the motion of the body be a simple translation or a rotation, *if only its*

radius is equal to the radius of inertia of the body. In any other case the acceleration produced is not proportional to the actual force itself, but to its equivalent at the radius of inertia. If we write k for this radius and r for the radius of any force f, then the equivalent force at the radius k will be $f_o = f\frac{r}{k}$, for we have just seen that for two forces to be equivalent, or to cause the same acceleration in a body, their moments about the virtual centre must be equal, or $f_o k = fr$, of which equation the one given above is only another form.

The algebraical forms of the propositions given above are as follows:—

(i) If f_1, f_2, f_3, &c., be the forces, then

$$f_1 + f_2 + f_3 + \dots\dots = 0 \qquad (1)$$

or more shortly

$$\Sigma f = 0 \qquad (2)$$

The sum (Σf) is not the mere arithmetical sum of the quantities, but their geometrical sum, taking their directions and positions all into account. If they are all parallel each one must be supposed to be *intrinsically* positive or negative, so that the sum would be

$$(\pm f_1) + (\pm f_2) + (\pm f_3) + \dots\dots = 0 \qquad (3)$$

(ii) and (iii). If f_1, f_2, f_3, &c., be again the forces, and r_1, r_2, r_3, &c., their virtual radii, while r is the radius of their sum,

$$r\Sigma f = 0, \qquad (4)$$

or

$$f_1 r_1 + f_2 r_2 + f_3 r_3 + \dots\dots = 0 \qquad (5)$$

Here the products or moments $f_1 r_1, f_2 r_2$, &c., are intrinsically negative or positive, according as the moment

tends to turn the body clock-hand-wise, or in the opposite sense.

If follows from equation (1) that

$$\left.\begin{aligned} f_2 + f_3 + \dots\dots &= -f_1 \\ f_3 + \dots\dots &= -(f_1 + f_2), \text{ \&c.,} \end{aligned}\right\} (6)$$

in words: **When the sum of the forces acting upon a body is *zero*, any one force, or the sum of any number of the forces, is equal and opposite to the remaining force, or the sum of the remaining forces.**

Further, if we have any forces f_1, f_2, f_3, of which the sum, f_0, passes through the virtual centre, then by definition

$$\left.\begin{aligned} f_1 + f_2 + f_3 &= f_0 \\ \text{or} \quad f_1 + f_2 + f_3 + (-f_0) &= 0 \end{aligned}\right\} (7)$$

If, therefore, to the given external forces acting on any body which is in static equilibrium, we add a force equal and opposite to their sum, the sum of the whole is *zero*, and we can apply to them the proposition stated above in connection with equation (6).

There exists an exactly similar relation among the moments of the forces. If follows from equation (5) that

$$\left.\begin{aligned} f_2 r_2 + f_3 r_3 + \dots\dots &= -f_1 r_1 \\ f_3 r_3 + \dots\dots &= -(f_1 r_1 + f_2 r_2), \text{ \&c.} \end{aligned}\right\} (8)$$

or in words:—**When a body is in static equilibrium, the moment about the virtual centre of any one external force acting upon it, or the sum of the moments of any number of such forces, is equal**

and opposite to the moment of the remaining force, or the sum of the moments of the remaining forces about the same point.

It will be remembered that these different statements as to the conditions of equilibrium do not describe different conditions, but only different aspects of the same condition, of which the most general statement was given in (ii). Which proposition is used in any particular case is generally a mere matter of convenience. In the following sections we shall give examples of the use of all of them.

§ 38.—STATIC EQUILIBRIUM OF PAIRS OF ELEMENTS.

We shall now take up the first problem alluded to in the last section (p. 268), but in the first instance only so far as it concerns pairs of elements, the simplest case in which it can occur. The problem is this: Given a body turning about a known point—its virtual centre—and acted upon by any number of known forces, f_1, f_2, &c., to find what force f must act in a given direction so as to keep the body in static equilibrium. The sum of the *known* forces we shall indicate by f_s in all cases, so that

$$f_s = f_1 + f_2 + f_3 + \ \ldots\ .$$

and the sum of *all* the forces, which must pass through the point O, we shall indicate by f_o, so that

$$f_o = f_1 + f_2 + f_3 + \ \ldots\ . \ + f = f_s + f.$$

In Fig. 104 we have a turning pair in the form of a disc and bearing. One known force only, f_1, acts on the disc; it is required to find f so as to bring the disc into

static equilibrium, or (as is often said) so as to balance f_1. Here f_1 and f are the only external forces acting on the body. Their sum f_0 must clearly pass through their join M, and it must also pass through the virtual centre O. The direction of f_0 must therefore be MO. To find f there is, then, nothing more necessary than to resolve f_1 in the

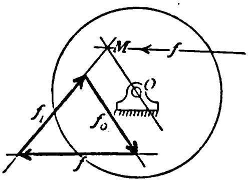


FIG. 104.

directions f and MO, that is, to construct a triangle of which one side is parallel and equal to f_1, and the other two sides parallel to f and f_0 respectively.

The sum, f_0, of all the external forces acting on the body, is therefore a single force passing through the virtual centre O, and the body is in static equilibrium. f_0 is in this case the total pressure of the pin against its bearing, and is balanced by the equal and opposite pressure of the bearing against the pin. This pressure is of course a force external to the pin, but we have seen (p. 260) good reasons for calling it a pressure rather than an external force.

In Fig. 105 the known force f_1 is a weight, and the unknown force f is the hand-pressure on a lever necessary to move it. The construction and lettering is precisely the same as in the last case. The lever, with its fulcrum (a cylindrical bearing), is, of course, simply a turning pair of elements, not differing in any way from the disc and bearing of Fig. 104.

In Fig. 106 the same case is given, but with two known forces, f_1 and f_2. Here we have first to find the sum of these two, f_s, and then to treat it exactly as we treated the single force f_1 in the last examples. The join of f_s with f is the point M, and through this point the sum of f_s and f—in other words the sum of f_1, f_2 and f—must pass, so that its direction and position are again given by the line MO. To find f we have only to resolve f_s in the directions of MO and of f as before.

It may very often happen that all or several of the forces coming into consideration in such problems as these are parallel. In this case the addition or resolution of the

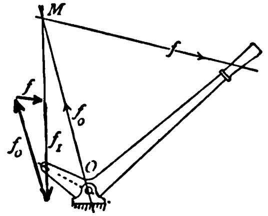


FIG. 105.

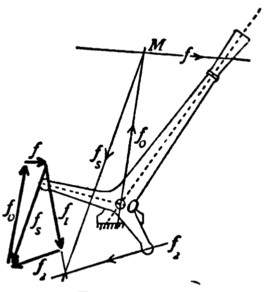


FIG. 106.

forces may require the use of the link polygon, and other graphic methods, with which the student should certainly make himself familiar. But there are many cases in which a simpler construction will suffice. We have seen that the only difference between the conditions stated in (ii.) and in (iii.) of the last section lay in the form of their statement. This difference, however, leads us to a corresponding difference of construction which is convenient in just these cases. An illustration is given in Fig. 107. A

lever is acted on by a force f_1, the force f to balance f_1 is required. The directions of f and f_1 are parallel, but in other respects the problem is of course identical with the ones we have just examined. The relation between the moments of the forces is given by the equation $f_1 r_1 = fr$, if r_1 and r be the radii of f_1 and f respectively. Hence $\frac{f}{f_1} = \frac{r_1}{r}$ and $f = f_1 \frac{r_1}{r}$, which is simply the algebraic form of the statement that the magnitudes of the forces, in order to keep the body in static equilibrium, must be inversely as their radii or "leverages." To find f it is therefore simply necessary to set off f_1 along the line of f, as at AB, draw the line BO . . ., and take for f the segment CD, which that line cuts off upon the force line f_1. For obviously $\frac{CD}{AB} = \frac{r_1}{r}$ whence $CD = AB\frac{r_1}{r} = f_1 \frac{r_1}{r} = f$.

Fig. 108 shows an exactly similar construction applied to the case where f_1 and f are both on the same side of the centre. In this case f_1 and f are of opposite senses, while if they are on opposite sides of O they have the same sense, the necessary condition being that their moments should have opposite senses.

In such cases as these, where f_1 and f are parallel, this construction is considerably simpler than the construction with the link polygon. It is not necessary that AB and CD should be drawn in the direction of the forces—they may be drawn in any convenient direction, as long as only they are parallel.

But there is no difficulty in reality in reducing these cases to exactly the former conditions. For the equality of moments has no connection with parallelism of forces, and there is no reason why we should not shift a force round

into any convenient position, if it is in an inconvenient one, so long only as we keep its radius unaltered. So, for instance, the lever of Fig. 107 might be treated as in Fig. 109, where the force f_1 is simply turned round into the position f_1', and the solution made precisely as in the former cases,

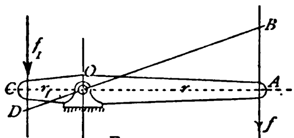


FIG. 107.

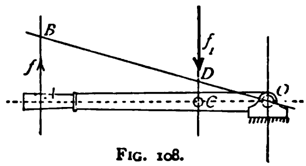


FIG. 108.

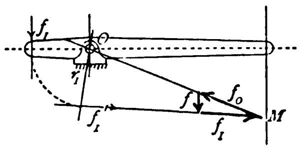


FIG. 109.

with of course precisely the same resulting value of f. It must be remembered, however, that the value of f_o, the pressure on the bearing, does *not* remain the same as before.

In Figs. 110 and 111 the body on which the forces are acting is an element of a sliding pair instead of an element of a turning pair. In the first case there is only one unknown force, in the second there are three, the case being that of the piston-rod of an obliquely placed cylinder.[1] The virtual centre O is in these cases at infinity, but as its direction is known, a line can always be drawn to it from any given point (as M) by simply drawing a line through

[1] The actual distribution of forces in such a case is much more complex than that sketched, the upward pressure of the guides, &c., being here left out of account. The position of the sum f_s in Fig. 110 has been found by link polygon construction, which is not shown in the figure.

the point at right angles to the direction of motion of the sliding pair.

The construction in all the examples given above may be summed up as follows :—

(i.) Find the sum f_s of all the known forces, by link polygon or any other construction, if necessary.

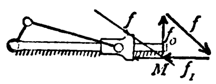


FIG. 110.

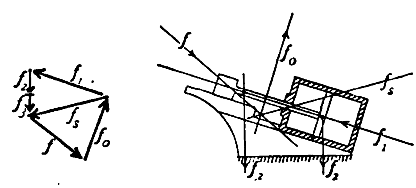


FIG. 111.

(ii.) Find the point (M) which is the join of f_s (or of f_1, if there be only one given force) with the direction of the unknown force f.

(iii.) Resolve f_s (or f_1, as the case may be) in the directions of f and of OM, the point O being the virtual (or permanent) centre of the body. The component in the direction f is the required force in that direction. The other component f_0 is the total pressure through the virtual centre, that is, the total pressure on the pin in Figs. 104 to 109, and the total pressure on the surface of the block in Fig. 110. This pressure (see p. 260) is balanced by the corresponding pressure or pressures of the body or bodies adjacent to the moving one, a matter which is looked at in detail in § 41.

In Fig. 112 is given an example of a different kind. One element of a turning pair is acted on by two forces, f_1 and f_2, which are equal and opposite to each other, that is, which form a couple. We require to find, as before, the force in the direction f which will hold the body in static

equilibrium. At first sight our construction appears to fail us here, for f_s, the sum of f_1 and f_2, is zero, and moreover, the join of f_1 and f_2 is a point at infinity. The difficulty, however, is merely apparent, not real. For we know that a couple may be shifted into any position without changing its action on a body as a whole.[1] We have only

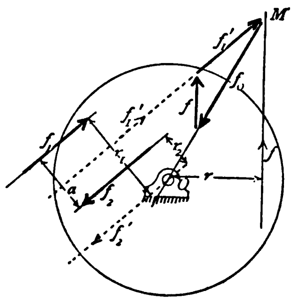


FIG. 112.

therefore to shift the couple, say to $f_1' f_2'$, so that one of its forces passes through the virtual centre (p. 265). This force will then be balanced by the pressures at that point, and will not further affect our problem. To find f we have only to take M as the join of f_1' and f, and resolve the former, exactly as before, in the directions of f and of MO.

We have to remember now, however, that f_0, the sum of f_1' and f, is no longer the whole pressure through the virtual centre, because we have now the external force f_2' acting there also. The pressure on the pin is therefore $f_0 + f_2'$. By the triangle we see at once that this is equal to f (f_2' being equal to f_1'). The counter-pressure exerted on the pin by its bearing must therefore be a force equal and opposite to

[1] The change of position will necessarily alter the distribution of stress in the material, but with that we are not here concerned.

f and acting through O. But such a force forms along with f a couple. This is a very striking example of the statement made in the last section in connection with equations (6) and (7). The body is under the action of four forces, f_1, f_2, f, and f_o, the last being a force equal and opposite to the sum of the other three, so that the sum of these four forces is zero. Therefore, any two of them must be equal to the remaining two; for instance: $(f_1+f_2)=(f+f_o)$. But $(f_1+f_2)=0$, for they are equal and opposite to each other, therefore $(f+f_o)$ must also $=0$, which they can only do if they also are equal and opposite to each other, and we have just found that this was actually their condition.

We have in our former construction applied the proposition that the moment of the sum of the forces about the virtual centre should be zero. We may equally well apply the proposition that the sum of the moments of the forces should be zero, by finding (graphically or otherwise) the moment of each force, adding all the moments together, and dividing by r, the radius of the unknown force f, to find that force. But this is seldom so convenient as the method already given. It is of special importance in its application to one case, however—the one, namely, where the sum of the forces acting on a body is a couple, as in the last figure. Here the sum of the moments of f_1 and f_2 is $f_1r_1-f_2r_2$ (f_1 and f_2 having opposite senses), and as $f_1=f_2$ this is equal to $f_1(r_1-r_2)$, which is the moment of the couple itself. If we write a for (r_1-r_2), the "arm" of the couple, we have therefore (in such a case as Fig. 112) $f_1a=f_2a=fr$ and $f=f_1\frac{a}{r}-f_2\frac{a}{r}$, which can be solved either arithmetically or by construction.

§ 39.—STATIC EQUILIBRIUM OF SINGLE LINKS IN MECHANISMS.

We have seen long ago that for all such purposes as we have at present in hand there is no difference between the motion of a link about a virtual centre and that of a lever or crank about a pin or shaft. So that all the constructions which we have given in the last section for pairs of elements apply equally well to links of chains, if the point O be taken as the virtual centre relatively to the fixed link. It will not be necessary to do more than give two or three examples to show this.

In Fig. 113 the forces f_1 and f act on the link a, which can turn about the point O_{ad}, here a "permanent" centre. The whole construction for finding f_0 is similar to that of Fig. 104, and similar lettering is used.

In Fig. 114 the forces f_1, f_2, and f act on the link b, which

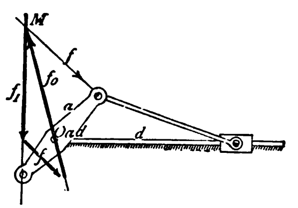


FIG. 113.

is turning about O_{bd}, a virtual and not a permanent centre. The construction is the same as in Fig. 106; f_1 and f_2 are added together to get f_s; the join of f_s and f, viz. the point M, is found, and f_s is then resolved in the directions of f and OM, the former component being the one required.

In dealing with virtual centres, however, it often occurs that they are inaccessible or inconveniently placed on the paper, so that either to draw lines through the virtual centre, or to find the length of virtual radii is troublesome. We have already in §§ 14 to 16 found methods by which in the determination of velocities we could dispense with the

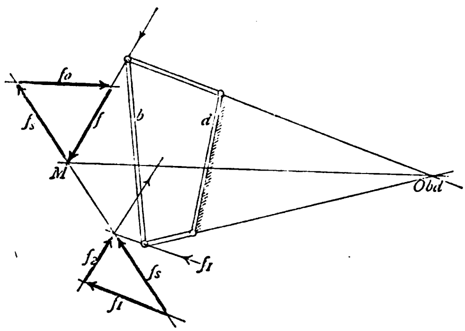


FIG. 114.

actual drawing of lines through the virtual centre if it were not convenient to draw them. But the velocities of points in a link vary directly as their virtual radii, and as the forces acting at them, if they are to be balanced, must vary inversely as *their* virtual radii, the forces must vary inversely as the velocities of points having the same radii, and must therefore be determinable by any methods which can be used for the determination of velocities.

Figs. 115 and 116 give illustrations of the applications of this method. In Fig. 115 the known and unknown forces f_1 and f act on the link b at the points B_1 and B respectively, and *in the direction of motion of those points.* To find f it is only necessary to set off f_1 from B along the

virtual radius of that point, that is, in the direction of the link *a*. Then through its end point R draw RP parallel to the virtual radius of B_1, that is, to the direction of the link *c*. RP (the point P being on the axis of the link *b*) is equal in magnitude to the required force f. The proof is simply (see p. 87) that by the construction $\frac{RP}{RB} = \frac{r_1}{r} = \frac{f}{f_1}$, (the virtual radii of the forces being equal to those of the points B_1 and B) and therefore $f = f_1 \cdot \frac{r_1}{r}$. The forces are therefore inversely as the virtual radii of the points at which they act, which is exactly the condition which we have already found that they ought to fulfil.

Fig. 116 represents the slightly more complex case where

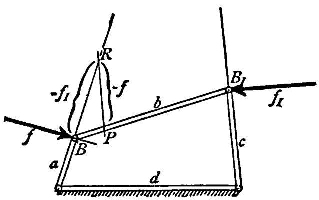


FIG. 115.

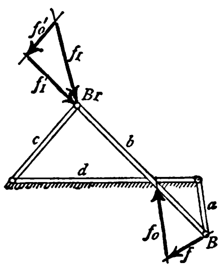


FIG. 116.

the directions of f_1 and f are *not* the directions of motion of the points on which they act. Forces may, of course, be assumed to act on a body at *any* point on their direction lines, but if the virtual centre is inaccessible it would be very inconvenient to have to find the particular point on each force line which is moving in the direction of the line. Without finding these points, however, the method of the last paragraph is not available. We can, however, use another

almost equally simple. Resolve f_1, the given force, into two components in the direction of the line $B_1 B$ and of the virtual radius of B_1 respectively. This gives us in f_1' the only component of f_1 which has to be balanced by f, for the remaining component f_0', acting through the virtual centre of b, does not require to be balanced by f. Next, as f_1' acts through B, the point of application of f, it is only further necessary to resolve it in the direction of f and of the virtual radius of B to find f, as is shown in Fig. 116. It is of course not necessary to carry f_1' along the line, as has been done in the figure for the sake of clearness. In practice the whole construction would be done at B_1.

It has been assumed that we knew the virtual radii of B_1 and B, the points at which the forces acted, which we may obviously do whether the virtual centre itself be accessible or not. At first sight it might seem that the problem was more difficult when it comes in the form of Fig. 117, but as a matter of fact the construction is practically the same. Here the forces, which are lettered as before, both act on the link a, the one at the point P_1, the other at the point P. If the virtual centre be inaccessible we cannot draw the virtual radius for either of these points.[1] But we must in all cases know the virtual radii of at least two points of the link, here the points D and B, the radii being the axes of the links d and b respectively. And the forces f_1 and f do not act at the points P_1 and P any more than at any other points along their direction lines, so that all we have to do is to treat them as acting at the joins A_1 and A of their direction lines with the known virtual radii. The line $A_1 A$

[1] There is, of course, no impossibility in drawing a line through P_1 or P in such a direction as to pass through the inaccessible join of the two other lines. But the construction for this is too long to be undertaken if it can be avoided.

takes the place of the line $B_1 B$ in the last figure, and the construction, as is shown, is exactly the same as before.

It is not necessary to give any examples when more than one force is known, for we have already seen that this involves no difference in principle; the known forces have only to be added together and their sum, f_s, takes in every respect exactly the place of the single force f_1 in the examples given above.

The case of a link under the action of a couple may just be mentioned; an example is given in Fig. 118. Its

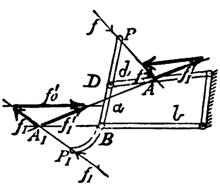


FIG. 117.

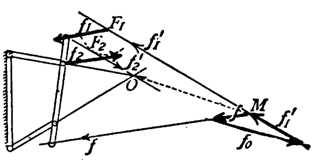


FIG. 118.

treatment does not differ in any way from that used with Fig. 112. We shift the couple until one of its forces passes through O, and then resolve f with the other exactly as in the former case. The easiest method of shifting the couple is that shown in the construction. From any point F_1 in the one force line draw a circle touching the other, and from O draw a tangent $O F_2$ to the circle. The new position of f_1 will lie through F_1 parallel to this tangent, as shown at f_1'. The point M will be, as in Fig. 112, the join of f_1' and the given line f.

Where the virtual centre of the link on which the forces are acting is at infinity (as the piston and rod in Fig. 111), the link has for the instant only a motion of translation, and we have seen that in such a case the action of a couple,

or of any number of couples, does not affect it in any way as regards static equilibrium (see p. 265).

Fig. 119 shows a mechanism interesting in itself as well as forming an illustration of the use of the method of moments described in the last section. The weight f_1 on b has to be balanced by a pressure f, the amount of which is required. We cannot carry out our former construction on the paper, because f and f_1 are parallel, and their join is therefore at infinity. We can draw a line parallel to them through O_{bd}, and the problem is then to resolve f_1 along the two given but parallel directions. There is of course no

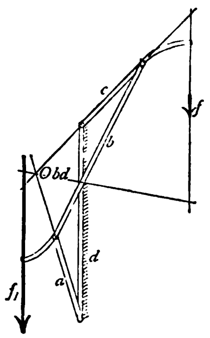


FIG. 119.

difficulty about this, but we may handle the problem as before —set off the given value of f_1 along the f line, and apply the construction of Fig. 107; we at once obtain the value of f. It is obvious that this mechanism gives us the means of gaining a very large mechanical advantage. For as O_{bd} is a virtual centre only, and not an actual pin joint, the distance of f_1 from it may be made as small as we please, and therefore the balancing force f may be made as small as we please.

If f_1 be made to pass through O_{bd}, it itself keeps the mechanism in equilibrium, and no force at all is required at f. If f_1 be shifted still further towards d, its moment about O_{bd} changes sign, and the force f, to balance it, must act *upwards*, not downwards. These changes made on a model (where of course the point O_{bd} is invisible) are very striking, and at first sight (like those described further on on p. 297) somewhat paradoxical.

Fig. 120, the mechanism of the common "knuckle joint,"

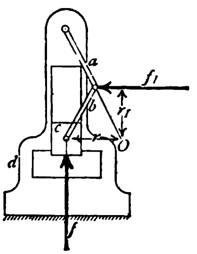


FIG. 120.

is an example of a somewhat similar kind. Here the link b is balanced under an equality of the moments $f_1 r_1$ and fr of the forces about the virtual centre O. As the link moves the point O changes, so that the arm r becomes continually less and less, and r_1 continually greater. Thus a given force f_1 can balance (or "exert") a greater and greater pressure f as the joint is pushed closer and closer "home," the property which we know to be characteristic of this mechanism.

§ 40.—STATIC EQUILIBRIUM OF MECHANISMS.

We have now to consider the general problem of the balancing of a known force, or forces, acting on any link or links of a mechanism, by a force given in position only on some other link. In the first instance, let us assume that the known forces all act on one link, and that the unknown force is to act on a link adjacent to it. We may suppose that in any case where several known forces are all acting on one link we can add them together, and substitute their sum or resultant for them, indicating it by the letter f with a suffix a, b, etc., according to the letter used for the link on which the forces are acting.

There is one very simple case, illustrated by Fig. 121, which does not involve any new difficulty even where the forces nominally act on non-adjacent links. For here the force f_a acts at O_{ab}, the point common to the links a and b, and the force f_c acts through O_{cb}, the point common to the links c and b. One of these points is a point in a and one in c, but both are also points in b, so there is no reason why we should not consider the forces as both acting on the same link b, and proceed to resolve as in Fig. 121, by finding the point M, the join of the two forces, and the point O, the virtual centre of the link b on which they act, and then resolving the given force f_c into components parallel to f_a (given in direction) and to the line OM. Such a case as this occurs very often indeed in practice. In the example given it represents the finding of the crank-pin resistance (f_a) balanced by a given piston pressure (f_c) in a direct-acting engine of the usual type.

A more difficult case is shown in Fig. 122, where a known force f_a acts on a, and has to be balanced by a force f_b

acting in the given direction on the adjacent link *b*. It is necessary in the first instance to find the three virtual centres corresponding to the two links acted on by the

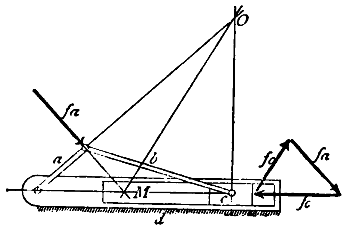


FIG. 121.

forces and the fixed link. Calling the fixed link *d*, these points are O_{ab}, O_{ad} and O_{bd};—we know that they must be on one line, and can at once mark their position in such a simple mechanism as this. It is supposed in this case that

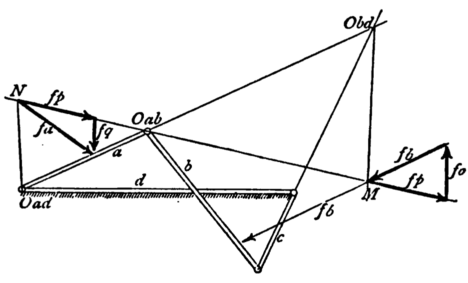


FIG. 122.

all three points are accessible. The next step is to resolve f_a into *any* two components of which one passes through O_{ad}, the fixed point of *a*, and the other through O_{ab}, the

point which a has in common with b. This may be conveniently done as shown in the figure, where f_p is the component acting through O_{ab}, and f_q the component acting through O_{ad}. We have thus at once converted f_a into an equivalent force acting on the link b. For by construction

(i) f_a is equal to $f_p + f_q$

(ii) f_q can have no effect on the motion of the body a or on the value of f_b, for it acts through the virtual centre or fixed point of a, and is therefore entirely balanced by pressures there. (It will be noticed that here and elsewhere f_q is not shown in its proper *position*, which is—here—along the line NO_{ad}.)

(iii) The only remaining component f_p acts through the common point of a and b, and therefore may be taken as acting on b direct.

The problem is now no more than that already solved in § 39. The given force f_p acting on a body b which turns about a known point O_{bd} has to be balanced by a force f_b acting in a given direction on the same body. The construction merely duplicates that of Figs. 113, 121, etc. Find M, the join of the two forces, and resolve f_p in the direction of f_b and of OM, the line joining M and the fixed point, O_{bd}, of the link b.

A very different example of exactly the same method is given in Fig. 123, an epicyclic train (see § 20), where a wheel d is fixed, and a wheel a carried round it by an arm b. There is given a weight f_a hanging from the periphery of a, and it is required to find the force f_b acting in the given direction on the arm b in order to balance it. The whole construction is identical with the last, and as the figure is exactly similarly lettered it is not necessary to repeat the statement of construction or proof.

The given force (or sum of given forces) f_a, can be

resolved through the required points in an infinite number of different ways—so that the point M can always be made accessible. But it may easily happen that O_{bd} is inaccessible. In that case it is necessary to choose such a point for M that we know the *direction* of MO_{bd}, even when the point O_{bd} itself is off the paper. This is done in Fig. 124, where

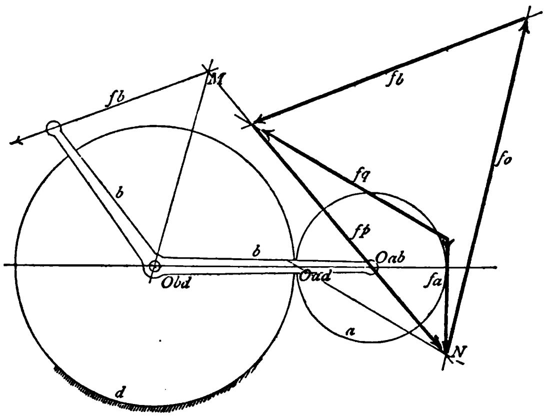


FIG. 123.

O_{bd} is purposely made inaccessible. The point M must lie on the line f_b, the given direction of the unknown force. If we take it at the point which is the join of that line and the axis of the link c, as has been done in the figure, we know the direction of OM, which is simply the axis of the link c. Drawing MO_{ab}, the point N on the force line f_a is fixed, as well as the direction of the component f_p. The given force f_a can then be resolved into components parallel to NM and to NO_{ad}, and the magnitude of f_p thus found. The resolution of f_p into f_b and a component f_o parallel to MO (*i.e.* to the axis of c) is exactly as in the former cases.

In Fig. 125 the same construction is applied to a slider-crank chain. Here the known force f_a acts on the link *a*, but the fixed link is *b*, and the balancing force which it is

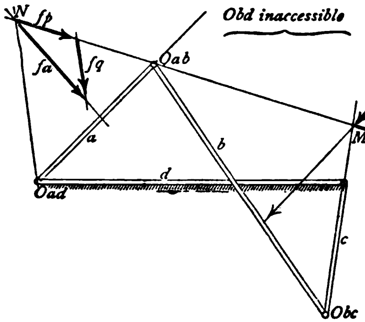


FIG. 124.

the object of the problem to find acts on *d*, the link adjacent to *a*. The point O_{bd} is inaccessible, but is known to lie on the line at right angles to the link *d* through the point O_{bc} (see p. 74). On this line, therefore, where it is

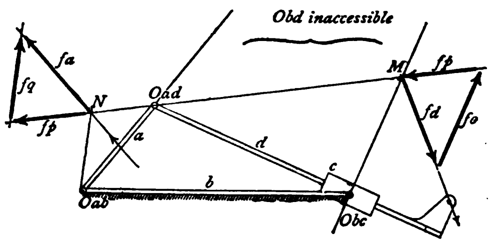


FIG. 125.

cut by the given direction of f_d, the point *M* is taken. *M* joined to O_{ad} gives the point *N*, and the construction follows as before.

This construction can, of course, be equally well applied when the virtual centre is accessible, and in practice it then

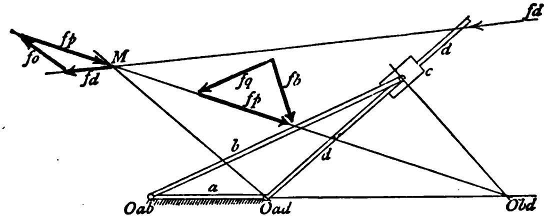


FIG. 126.

forms an excellent check on the results obtained in the more general manner of Fig. 122, where the point *M* was allowed to take any position in which it came.

In most cases it is as easy in fact, and in all cases as simple in theory, to determine the equilibrium of a mechanism under forces on non-adjacent as on adjacent links.

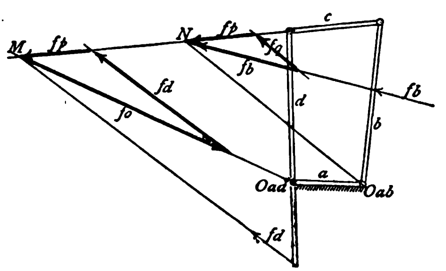


FIG. 127.

Figs. 126 and 127 give illustrations of this. The link *a* is taken as fixed in each case, and the known force is

f_b acting on the link b, the unknown force f_d acting on the link d in the given direction. The three virtual centres to be determined are therefore O_{ab}, O_{ad} and O_{bd}. The force f_b is first resolved into two components f_p and f_q, the latter passing through O_{ab}, the fixed point of the link b, the former through O_{bd}, the point which b has in common with d. The point M is then taken as the join of f_p with the given direction of f_d, and f_p is resolved along $O_{ad}M$, (that is, through the fixed point of d), and along the direction of f_d, the magnitude of which force is thus found. In Fig. 127 the point O_{bd} is inaccessible, and the same construction is used as in Figs. 124 and 125.

It will be noticed that the case is entirely unaffected by the fact that there is a third link c (there might be half a dozen) between b and d. The existence of such a link has had its effect in fixing the position of the point O_{bd}, but this point having been once found it may be dealt with exactly as if b and d were (for the instant) pivoted together at it, quite independently of the form of the actual pairing or linkage which has taken the place of that direct connection.

It is obvious that the first step in this construction is the finding of three virtual centres, viz., the fixed points of the two links on which the forces are acting, and the common point of the same two links. The two former (here O_{ab} and O_{ad}) are generally accessible, but when the chain is a compound one, and b and d not links adjacent to the fixed link, they may be inaccessible. The third virtual centre may in many cases be inaccessible. The drawing of lines through O_{bd} may be dispensed with by the method given above. The necessity of drawing through the other two points may often be dodged by the sort of construction used in Figs. 40 to 42, etc. In a case such as Fig. 127, the inaccessibility of O_{bd} has made us use the axis of the link c

as a line passing through it, and it may of course happen that the join of c with f_d, (the point M), is thereby made inaccessible. In such a case we may use the method of Fig. 109, and turn f_d round O_{ad} through any angle (keeping only its distance from that point unchanged), until M comes into some convenient position. It is very seldom that any direct construction for drawing lines through inaccessible points requires to be used.

We have seen that the method of resolution given applies when the mechanism is balanced by forces on non-adjacent as well as on adjacent links. Obviously it makes no difference whether the mechanism is simple or compound (p. 68), although in the latter case the finding of the virtual centres is necessarily more troublesome. Without their use, however, the problem is so complex that it would hardly be attempted at all. Before giving examples of applications to compound mechanisms it will be well to state the problem and solution in its general form, applicable alike to the simplest and the most complex case.

Given a mechanism of which r is the fixed link, and s and t any other two links, given also a force f_s acting on the link s, to find the force f_t acting in a given direction on the link t which will keep the mechanism in static equilibrium.

(i.) Find the three virtual centres O_{rs}, O_{rt} and O_{st}, which must (§ 12) be three points in one line.

(ii.) Resolve f_s into two components, of which one, f_q, passes through O_{rs}, and may be neglected, and the other, f_p, through O_{st}.

(iii.) Find the point, M, when f_p joins the given direction of f_t, and resolve f_p into two components, of which one is in the direction MO_{rt}, and may be neglected because it passes through O_{rt}, and the

other is in the given direction of f_t, and is therefore the force required.[1]

We shall now proceed to deal with a few cases of mechanisms which present, in reality or in appearance, difficulties somewhat greater than those of the examples already given. The lettering will be in all cases the same as that just given in the general statement of the solution of the problem—*r* a fixed link, *s* and *t* the other two links, *s* being the one on which the known force acts. The construction for finding the virtual centres is not given, as it has been so fully discussed in § 12.

In Fig. 128 is shown the Peaucellier parallel motion, which will be more fully examined later on (§ 47). The given force f_s is supposed to act on the particular point which moves in a straight line, and to be in the direction of its motion. It is required to find what force f_t in the same direction will balance it, acting on what may be called the crank pin at the end of the link *t*. The construction used requires no explanation, but it is a noticeable point about the example that the force f_s acts at the common point of the links *s* and *s'*, the force f_t at the common point of the links *t*, *t'* and *t.*" f_s may therefore be supposed to be acting on either of two, and f_t on either of three, links. It will be a good exercise to go through the corresponding six solutions, and see that they all give the same result, as necessarily they do.

[1] In a paper read before the London Mathematical Society in May 1878, and published in Vol. ix. of their *Proceedings*, the author has given another equally general solution of this problem by a method of "virtual mechanisms." The solution given here is, however, not less general, and is somewhat simpler. He has therefore not thought it necessary to give also the former, which he believes to have been the first general solution of the kind published. It need hardly be pointed out that, although the words mechanism and links are used, for obvious reasons, in the statement given above, the solution is perfectly general in respect to the static equilibrium of any three bodies having plane motion and acted on by forces in one plane.

Fig. 129 illustrates the case of a compound epicyclic train. A known weight f_s hangs on the free end of the arm

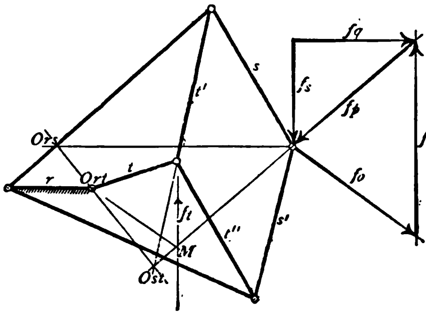


FIG. 128.

s; it is required to find what pressure f_t at the pitch radius of the wheel t will balance that weight. The three virtual centres are found as on p. 157, the whole train being equivalent (as will be remembered) to the pair of wheels (one annular) of which portions are shown dotted. There is nothing new about the construction, which need not be commented on. But it is interesting to notice the change in f_t with change of position. If, for instance, it were acting in the position f'_t it would be not an unnatural guess at first sight that it would be much *less* than before, its radius from the centre of the wheel (O_{st}) being so much increased. Its distance from the centre of the wheel has, however, nothing to do with the matter, for the wheel is not turning about its own centre at the instant,[1] but about the point O_{rt}. And the new position is not only nearer to O_{rt} than the former one, but is on the other side of it, so that the new value

[1] That is, its motion about its own centre forms only a *part* of its whole motion relatively to the fixed link r.

f'_t is not only greater than before, but is in the opposite sense. To balance a weight f_s it must act *upwards*, not *downwards*. If f_t had acted actually through the point O_{rt} it would have been impossible for the mechanism to be balanced in static equilibrium by it, any more than a weight hanging on a pulley could be balanced by a force acting through the axis of the pulley shaft. In the latter case, however, the matter looks perfectly simple, whereas here it has almost the

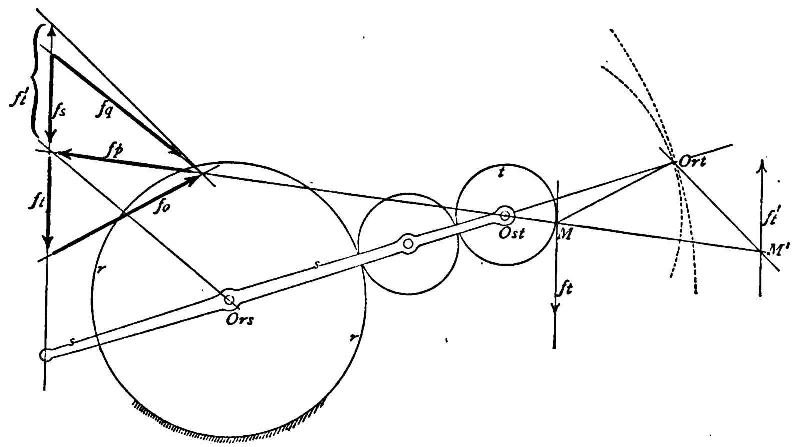


FIG. 129.

appearance of a paradox to construct this mechanism as a model, attach a sufficiently long arm to the wheel t, and then to see how a weight hung on the arm becomes less and less effective as it is moved outwards, and finally helps, instead of resisting or balancing, the motion which the weight f_s tends to cause.

The case of Fig. 130 represents the first part of the mechanism of some "differential pulleys." It differs from Fig. 129

only in the fact that one of the wheels is annular. The arm t takes the form of a rope wheel, with a radius equal to the radius of the force f_t. The weight is supposed to hang from the wheel s at the radius of the force f_s, and it will be seen that the combination, even as it stands, has a very considerable mechanical advantage, represented by the ratio $\frac{f_s}{f}$, which in the figure is $= 6$.

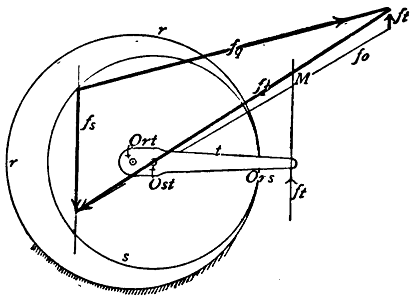


FIG. 130.

In § 20 "reverted" wheel trains were examined. If it were attempted to treat such a train in its actual constructive form by the methods of this section, we should find that we got no result, for in many cases the three virtual centres with which we work would fall together in one point—as has been seen in the section referred to. But this need not in reality interfere with the solution of the problem, because we have seen that the reverted form was used purely as a matter of convenience, and that a reverted train differs neither kinematically nor mechanically from a wheel train with the same sized wheels, having their axes spaced out in the usual way. Hence to work out the problems of this

section with a reverted train, such as Fig. 69 of § 20, we have only to alter it into the equivalent form of Fig. 66, and it falls at once into the cases we have just been examining. It is only necessary to remember that in shifting a link on which any force is acting the force must be shifted with it, so as to have the same virtual radius in the new that it had in the old position.

One illustration of this must suffice, the driving mechanism of a capstan (Fig. 131). Here the capstan head is attached

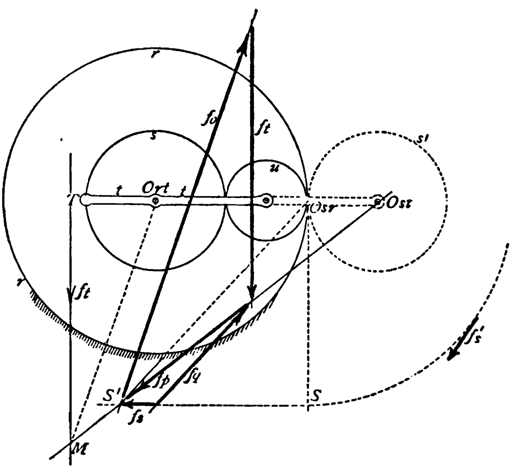


FIG. 131.

to a shaft on which the wheel *s* is fixed. As this rotates it compels the rotation of the idle wheel *u* between its own periphery and that of the concentric fixed annular wheel *r*. The spindle of *u* is carried in the bottom of the rope or chain drum, which itself can turn independently about the same axis as that of *s* and *r*. The drum therefore forms the

arm $t\,t$ of the train, and the whole mechanism is a reverted epicyclic train of three wheels, one (forming the fixed link) an annular wheel, and one merely an idle wheel. A known force, f_s, acts on the capstan head at a known radius—(this would be naturally the pressure on one of the capstan bars)—it is required to find what resistance it can balance on the drum or arm t at the radius of the point T. In the first place we must substitute the wheel s' for s, for the reason already given, then mark the points O_{rt}, O_{sr} and O_{st}.[1] After this the construction proceeds precisely as before. It will be noticed that even with the small radius at which f_s is supposed to act the mechanism has a very large mechanical advantage (8 : 1 in the figure), which is enormously increased in practice by applying f_s at the end of a very long bar. It must be specially noticed that as the direction chosen for f_s has been parallel to the line along which the wheel s has been shifted, it has not been necessary to alter it, for f_s has the same radius in reference to the new as to the old axis of the body (s) on which it acts. But there is no necessity whatever that f_s should be taken where it is. Such a position as f'_s would give precisely the same results, if only the new line lay at the same distance from O_{sr} as the old one.

If instead of a single force, or a series of known forces of which it is the *sum*, acting all on one link, we had to deal with a known force or forces, each acting on a *different* link, it will be found most convenient to determine the value of the balancing force separately for each force or set

[1] It happens that in this case the idle wheel goes out of account altogether when the wheel s is put into its new position, for an externally toothed spur wheel r would constrain exactly the same motion in s^1 *directly* that the annular wheel would through the intervention of the idle wheel. But this, and the particular position in which O_{sr} comes, is accidental to the special form of mechanism examined.

of forces, and then add together all the quantities so found.

For instance, let there be given a force f_a acting on a link a (Fig. 132), f_b on link b, and f_c on link c, and let the problem be to find the force f on link c, in a given direction, necessary to balance all three. Nothing more is necessary than to resolve separately each force which is not already acting on the link c, into two components, one acting through its virtual centre relatively to the fixed link

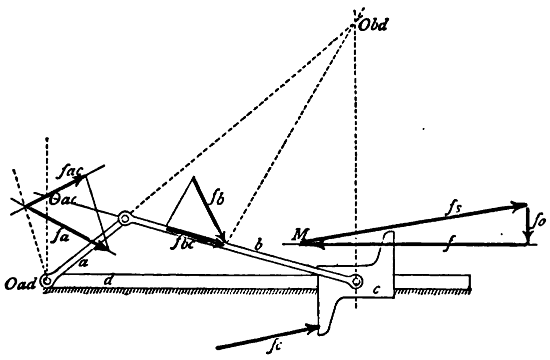


FIG. 132.

d (and therefore to be neglected), and the other acting through its virtual centre relatively to the link c. The components acting on c may then all be added together and treated as one force f_s, from which f can at once be found.

In the figure the components of f_a and f_b acting on c are lettered respectively f_{ac} and f_{bc}. These together with f_c are found (by link polygon construction omitted in the figure) to be equal to f_s. The join of f and f_s is M, and the magnitude of f is found just as formerly.

It is obvious that the values of the components f_{ac} and

f_{bc} will differ altogether according to the particular resolutions of f_a and f_b adopted. But if their magnitude varies, their position and direction vary correspondingly, and their sum, or balance, f, remains unchanged. In the general case, where the link c has its virtual centre, O_{cd}, relatively to the fixed link at a finite distance, it is the moment of each component about O_{cd} which remains unaltered. In the present case, where O_{cd} is at an infinite distance, it is the components of f_{ac}, &c., in the direction of motion of c, which remain unchanged.

The differential pulley block, of which a part has been already examined in Fig. 130 forms an excellent example of a case exactly similar to the last in reality, but much more puzzling in appearance. In Fig. 130 it was assumed that the annular wheel r was stationary. In reality, the pulley is ar-

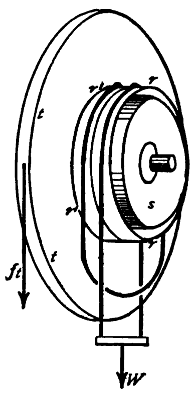


FIG. 133.

ranged as shown in Fig. 133. There are two separate annular wheels r and r^1, placed face to face, and together forming a box, within which the wheel s moves, this wheel being broad

enough to gear with both of them at the same time. Each wheel carries outside it a drum, and from each drum hangs one half of the chain suspending the weight. Neither of the annular wheels is fixed, they are free to revolve, each with its own chain drum, about the same axis as that of the rope wheel or arm *t*. If now *r* and r^1 were the same size they would be caused to revolve slowly, but both at the same rate, by the motion of *s*. The chain would be lowered on one side precisely as much as it was raised on the other, and the weight would remain stationary. But one wheel is made a little larger than the other,[1] and they consequently rotate at slightly different velocities, and the chain is a little more wound up than lowered, or *vice versâ*. The problem is, what force must be exerted at the periphery of the wheel *s* in order to lift a given weight at the radius of the chain drums. The three wheels *r*, *s*, and *r'* form together a reverted wheel train, which in order to be dealt with must have its axes placed out as in Fig. 134. The wheel *r* has 44 teeth, *r'* 45, and *s* 40 teeth, but the differences between them have been somewhat exaggerated for the sake of clearness in the engraving. The problem is exactly the one treated in connection with the figure 132. On *r* and *r'* act equal forces f_r and f'_r at equal radii, and tending to turn them in opposite senses, *i.e.* to balance each other; required the balancing force on *s*. By resolving each force through the fixed point of the link on which it acts and through the common point of that link and the link *s*, exactly as before, we get f_s, acting almost through the centre O_{st}.[2]

[1] To all appearance the wheels are the same size, but one has one tooth more than the other. As the teeth of both are of the same pitch, both gearing with *s*, it follows that the pitch diameter of one is, in just the ratio of the number of teeth, larger than that of the other.

[2] If the radii of *r* and r^1 were equal, the direction of f_s would pass through O_{st}—that is, no force would be required on *s*—which would be

At any other position on *s* the balancing force will be inversely proportional to its radius, for instance, at the periphery of *s* the force will be $\frac{1}{45}$ of f_s, a quantity too small to be clearly shown on the figure.

The working of this mechanism is generally explained by referring to its velocities. In consequence of the "odd

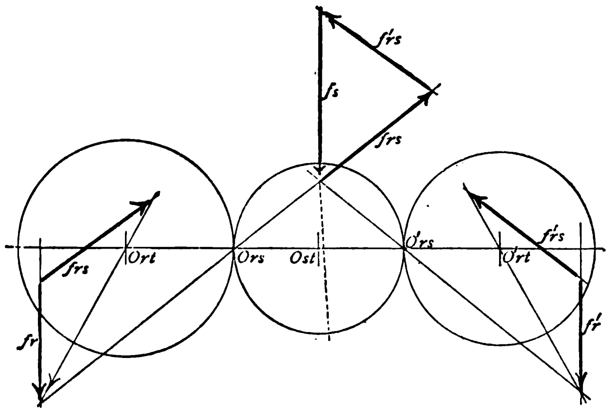


FIG. 134.

tooth" or "hunting cog" the one wheel makes (say) $\frac{45}{44}$ times as many revolutions in a given time as the other. The weight is lifted then in the same fashion as if there were spur gearing of 45 : 1, and the resistance and effort are in the same ratio. But this explanation, although correct so far as it goes, is so vague, and helps so little to the understanding of any other similar cases when they occur, that it has been thought worth while to examine the working of the mechanism in

already balanced, but in that case of course no weight would be lifted. (The dotted line in the Figure ought to be in line with f_s.)

detail, and see how it could be handled by such methods as we had found sufficient in other cases.

It may be worth while, in concluding this section, to point out once more that the force problems which we have just been considering are essentially the same as the velocity problems examined in Chapter V., when, as here, we leave out of consideration the masses of the bodies acted upon, and assume that no accelerations exist, and no frictional resistances. If a mechanism be balanced under two forces acting on different links, the magnitudes of those forces must be such that their components in the direction of motion of the points at which they act must vary inversely as the velocities of those points, exactly as if they acted on one and the same link or body. The methods we have adopted enable us to arrive at the result without actually referring to their velocities or apparently determining them. They are nevertheless necessarily involved in our constructions, and one example may be given to show how, if necessary, they could be made use of. We may take the capstan (Fig. 131) for this purpose. If we suppose the forces f_s and f_t acting on S and T respectively, as we are entitled to do, their magnitudes should be inversely as the linear velocities of those points. But by § 15—

$$\frac{\text{Linear vel. } T}{\text{Linear vel. } S} = \frac{\angle\text{r vel. } t}{\angle\text{r vel. } s} \times \frac{O_{rt}T}{O_{sr}S}$$

the last-mentioned quantities being the virtual radii (relatively to the fixed link) of the points at which the forces were acting. From § 20 we know that

$$\frac{\angle\text{r vel. } t}{\angle\text{r vel. } s} = \frac{1}{1 + \text{vel. ratio of train}} = \frac{1}{3 \cdot 2}$$

and by measurement we find the ratio of the virtual radii

$$\frac{O_{rt}T}{O_{sr}S} = \frac{2}{5}.$$

Hence linear vel. $T =$ linear vel. $S \times \frac{1}{3 \cdot 2} \times \frac{2}{5}$ = linear vel. $S \times \frac{1}{8}$, and correspondingly $f_t = 8f_s$, as in the figure.

If S had been taken at any other point in the force line, as S', with virtual radius $O_{sr}S'$, such a calculation as has been above made would give the value of the component of f_s at right angles to that radius (namely, in the direction of motion of the point S'), and from that component of course f_s could be found. But it is most convenient to work with points, as S and T, whose direction of motion is the direction of the forces acting on them; unless they are inaccessible.

§ 41. STATIC EQUILIBRIUM OF FIXED LINKS IN MECHANISMS.

The problems occurring in connection with the fixed link of a mechanism are generally of a somewhat different nature from those we have just been considering, but so far as we are here concerned they are very simple. Forces can only be transmitted to the fixed link through the links adjacent to it—forces acting on non-adjacent links must be transmitted through the adjacent ones. Moreover, each of these adjacent links has only one point in common with the fixed link, so that all forces have to be transmitted to the fixed link through the points which are the virtual centres of the adjacent links relatively to it, as for example the points O_{ad}, and O_{cd}, in Fig. 135.

As to these forces themselves, they are simply the pressures forming the components which we have been neglecting. We have resolved each force into two components, one acting through a virtual centre (which we have neglected) and the other capable of causing acceleration, and therefore requiring to be balanced. So far as moving links are concerned this latter component is the only one which we had to consider. So far as the fixed link is concerned, on the other hand, it is only the former that have to be attended to. Every one of them passes through some determinate point in the fixed link, *i.e.* through some fixed point, because it passes through the point which the moving link has in common with the fixed one. The magnitude of each is determined. The fixed link therefore is nothing more than a body in equilibrium under a number of completely known forces (pressures) acting through completely determined points, balanced by

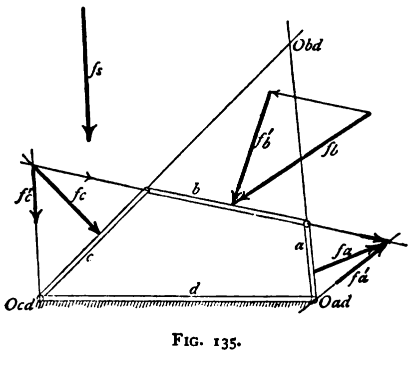


FIG. 135.

corresponding pressures at the different fastenings, which prevent the fixed link moving relatively to the earth, which, in fact, fix the link.

Fig. 135 shows a simple mechanism in equilibrium under

three forces, f_a on a, f_b on b and f_c on c. The equilibrium between these forces we may suppose to have been already determined, and the components through the virtual centres O_{ad}, O_{bd} and O_{cd}, which are respectively f_a', f_b', and f_c', are drawn. These three forces must now be added together (the construction for this is not shown in the figure) and their sum f_s found, which is therefore the sum of all the forces acting on the link a.

As there are only two links, a and c, adjacent to the fixed link, f_s must actually be received by the fixed link in the form of pressures acting on it from these links, and must therefore be resolved into two forces, acting through O_{ad} and O_{cd} (under the conditions described below) in order to find the actual forces tending to move the fixed link.

There is no reason, except as a matter of convenience, for resolving as we did in the last section, by taking one component of f_c along the direction of b. With the particular mechanism of the figure it is handy to do so, that is all. All that is necessary is to resolve each force which is not already acting at a virtual centre into components which are acting through virtual centres, one of which centres must be the fixed point of the moving link. Each different method of resolution will of course give quite different values of f_a', f_b' and f_c', but it will leave their sum f_s unaltered, and therefore the actual pressures on the fixed link, through the links adjacent to it, will remain unchanged, which is all that we are concerned with.

We may now assume that we can find the forces or the sum of the forces acting on the fixed link, and there remain only to consider any special questions arising out of the resolution of this sum into the various supporting or fixing forces. We notice at once that we are here in a position different

from our former one, for the fixed link has no one particular fixed point—no virtual centre relatively to itself, in fact, but is fixed at all points. Hence we have no longer available any constructions which depend on the use of such a point.

It is not only the fixed link of a mechanism which comes into this case, but any body acted upon by a system of forces under which it cannot, or must not, move at all. So far as we have hitherto gone, we have considered the conditions under which the movable links of a mechanism could either be kept stationary or caused to move with a uniform velocity. The latter alternative we have now to eliminate, and to consider cases where the body must be stationary, at least so far as the particular forces considered are concerned. The crane post of Fig. 137 must be able to revolve, but is nevertheless a stationary body so far as all forces in the plane of the paper are concerned, and the same thing is true of the shaft and wheel of Fig. 139. The bracket of Fig. 140, on the other hand, is a body which must be stationary under every force which can possibly act on it.

The condition of equilibrium is now that the sum of all the external forces must be zero, and the sum of their moments not only about the virtual centre, but about any point in the plane also zero, as in § 37. But here there is no longer any possibility of balancing disturbing components by a force acting through the virtual centre. Hence it is only in some cases possible to fix beforehand the direction of the force whose magnitude has to be determined, as in the problems we have hitherto dealt with. Its direction, as well as its magnitude, is fixed by the given forces, and the former is known beforehand only in some special cases.

Fig. 136 represents a case which may occur in practice.

A frame which is acted on by three forces, f_1 f_2 f_3, forms a fixed link. The points by which it is fixed are A and B. The magnitude and direction of the fixing forces f_a and f_b are required. In the first place, f_1, f_2, and f_3 are replaced by their sum f_s, whose magnitude and position are found in the usual way. The problem then is to find two forces which, acting through A and B, will be equal and opposite to f_s. It is obvious that for the directions of f_a and f_b, any two lines meeting on f_s, such as those shown, will suffice, and to find f_a and f_b it is only necessary to resolve f_s in these two directions. It is clear that both these directions cannot be fixed

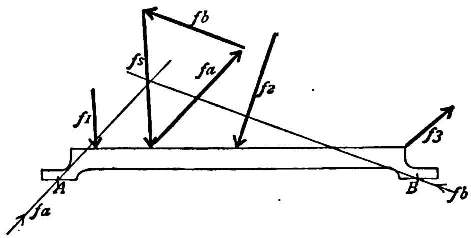


FIG. 136.

beforehand, as their point of intersection must lie upon f_s; *one* of them, however, may be so fixed. It is clear also that the body could not be kept in equilibrium by any *one* fixing force alone, unless that one were exactly opposite to f_s, and could change its position with every change in the other forces. With the *two* fixing forces this change can of course occur without difficulty, as the force lines (or the one of them which is not fixed), can turn in any fashion and to any angle about their fixed points A and B, if only the nature of the fixing (bolts, keys, etc.) is such as to afford sufficient stiffness in all possible directions of the forces. Further, if

the fixing points are three in number, the forces may, and if they are more than three the forces must, become statically indeterminate.

Figs. 137 to 140 are illustrations of the much more important cases where the body is kept fixed by *couples*, and where also it is generally possible to fix beforehand the directions of all the forces. Let the body, for example, be a crane post, the load on the crane being f_1. By the construction of the crane the only forces which can possibly act

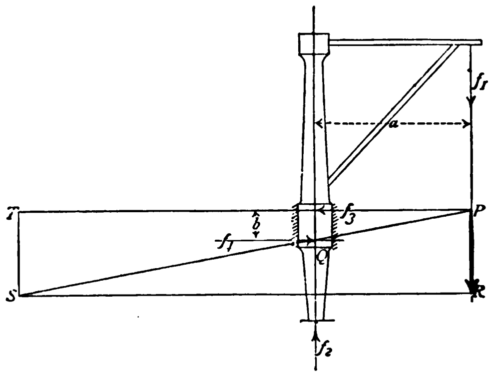


FIG. 137.

on the post are f_2, f_3, and f_4, and by the nature of the connections f_2 must be vertical, and the other two horizontal, and therefore at right angles to the former.[1] It follows that f_2 must be equal to f_1, so that together they form a couple, whence f_3 and f_4 must also form a couple (§ 38). The moment of the two couples must be equal, so that $f_1a = f_3b$, a and b being known dimensions; the unknown force is thus

[1] Friction is here disregarded, of course, as in all our present problems. Its effect is examined in Chapter XII.

determined at once as $f_2 = f_1 \frac{a}{b}$. The graphic solution is of equal simplicity, and is based on Culmann's method of resolving a force simultaneously into three components. Call the join of f_1 and f_3, P, and that of f_2 and f_4, Q. Join PQ. Set off $f_1 = PR$ from P, and from its end point R draw a parallel to f_3 and f_4. RS is then the value of f_3 and f_4. This is just equivalent to resolving f_1 into the directions of f_3 and of PQ, and then resolving the last-named component in the direction of f_2 and f_4, as ST and TP. That the construction is right also follows at once from the evident equality $\frac{ST}{SR} = \frac{b}{a}$, whence $SR = ST\frac{a}{b} = PR\frac{a}{b} = f_1\frac{a}{b} = f_3$.

An exactly similar calculation or construction will determine the forces f_5 and f_6 (again forming a couple) which

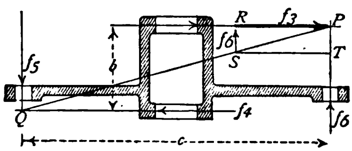


FIG. 138.

balance f_3 and f_4 in the bedplate casting of the crane, Fig. 138. Here $f_5c = f_3b$, whence $f_5c = f_1a$, and $f_5 = f_1\frac{a}{c}$, so that f_5 could be found irrespective of f_3 if required. The construction for finding f_5 from f_3 is given in Fig. 138. The lettering corresponds to that in Fig. 137.

It will be seen that in both these cases the construction can be proved by the method of similar triangles as well as by the proof given. For $\frac{SR}{PR} = \frac{a}{b}$, so that $b.SR = a.RP$, or $bf_3 = af_1$, which was required.

Fig. 139 is another example of a case which occurs in practice. A force f_1 acts, as shown in the Figure, on an overhung spur-wheel fixed upon a shaft resting in two bearings. It is required to find (i) the pressures f_2 and f_3, caused by the wheel on the shaft, and (ii) the supporting forces or pressures f_4 and f_5.[1] This involves nothing more than the resolution of f_1 first into components along f_2 and f_3, and then

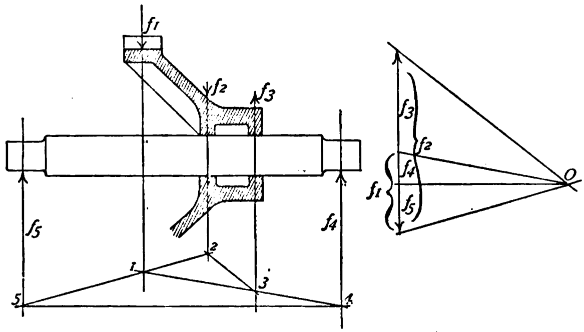


FIG. 139.

into components along f_4 and f_5. The construction is that usually employed for the resolution of a force into two components parallel to itself. Through any point 1 in f_1 any two lines are drawn, and these give at once points 2 and 3, and 4 and 5, by joining which the link polygons are completed. f_1 is drawn in the force polygon, and by drawing rays parallel to 13 and 15, the pole O is found at their intersection.

[1] Both here and in the case of the crane post the bodies supposed fixed can be moved. But in each case the particular forces dealt with are such as could not move them, because they all act through or parallel to their virtual axes (that is, in the plane of the paper). The bodies are therefore, so far as our present problems are concerned, fixed.

Parallels to 23 and to 45 give at once the values of the required forces f_2 to f_5.

Fig. 140 represents a case very often occurring in practice. A bracket is loaded by a known force f_1 in a known direction, and supported by horizontal forces at f_2 and f_3 and by a vertical pressure at f_4. The values of f_2, f_3, and f_4 are required, the bracket being in static equilibrium. The most

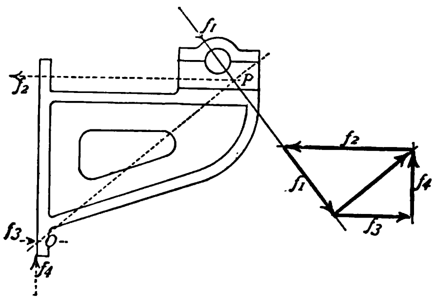


FIG. 140.

convenient construction is again that given on p. 312 above, for the simultaneous resolution of a force into three components. Find P the join of f_1 and f_2, and Q the join of f_3 and f_4. Resolve f_1 in the direction of f_2 and PQ, and then resolve the component parallel to PQ in the directions f_3 and f_4, as shown in the force polygon; this gives all three forces at once. Had f_1 been vertical, the construction would have been exactly similar, but in that case f_4 would have been equal to f_1 and f_2 to f_3, and the figure would have been a rectangle exactly similar to some of those already examined, and the forces acting on it would have been a pair of couples.

§ 42. POSITIONS OF STATIC EQUILIBRIUM.

There is a class of problems sometimes of importance in connection with mechanisms which can very easily be solved by the help of the virtual centre, and which may be looked at here. A mechanism is acted on by some unbalanced force—(or force balanced only by resistance to acceleration, see p. 266). Leaving aside all questions as to acceleration while the mechanism is in motion, it is required to find in what position it will be when the force brings it to rest, or whether the force can ever bring it to rest at all.[1]

The problem is solved by finding the position occupied by the mechanism when the force comes into line with the virtual centre of the link on which it acts (if it ever does take that position), for we know that motion must cease when the force producing it acts through the virtual centre. In complex cases this position may have to be found by trial, but in many instances it can be found at once. Of the simpler cases there are three kinds, at each of which we may look, viz. (i) when the force remains unchanged in position (relatively to the fixed link) as the body moves, and acts therefore at continually changing points of the body, (ii) where the force acts always through the same point of the body and remains always parallel to itself as the body moves, (iii) when the force acts always through the same point of the body and remains in the same position relatively to the body, so as continually to change its position relatively to the fixed link.

In the case of a pair of elements a force under condition

[1] The following constructions were mostly given in a paper read by the author before section *A* of the British Association at its Glasgow meeting in 1876.

(i), (as *e.g.* in Fig. 141), will cause motion continually as long as it lasts, for as the virtual centre is a permanent one the force will never pass through it if it does not do so to start with, and if it does so to start with, the body will not

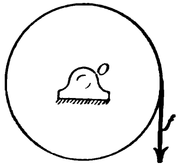


FIG. 141.

move. But if the force be acting on a link in a mechanism, such as *c* in Fig. 142, whose virtual centre changes, then it

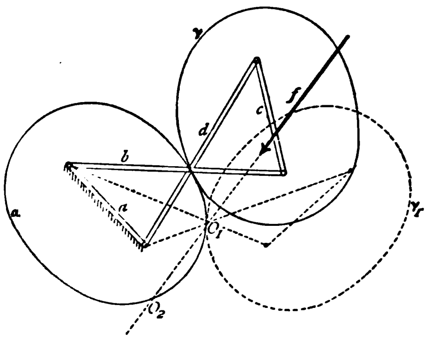


FIG. 142.

may or may not bring the mechanism to rest. In Fig. 142 the curve *a* is the centrode of *a*, the fixed link. The force line *f* cuts this curve in two points O_1 and O_2. The mechanism will therefore come to rest in the position

shown by the dotted lines, where O_1 has become the virtual centre O_{ca}. Of course such extended form is pre-supposed for the link c as will allow f to continue acting on it in its altered position. If the force (as in this case) cuts the centrode in two points, it causes *stable* equilibrium only at one of them (*i.e.* the one *from* which the force appears to come to the other), here O_1. The other point is one of *unstable* equilibrium. If the force be actually acting through O_2 it will tend to keep the body stationary, but if it acts in the least on either side of O_2, it will turn the body round in one sense or the other until O_1 becomes the virtual centre. If motion continue a small distance beyond O_1 the force f will tend to bring the mechanism back again, not to carry it forward. A reversal of the sense of the force would, of course, make O_2 the stable and O_1 the unstable centre. The whole process will perhaps be better followed if it is looked upon as the rolling of the centrode γ upon the centroid α, for we have already seen (§ 9) how this rolling necessarily accompanies the motion of the mechanism. If f did not cut the curve α there would be no position of equilibrium for the link under that force.

Fig. 143 illustrates case (ii), where a force remaining parallel to itself acts always through the same point of a body. The moving body is here one element in a turning pair. The body will move until the point F occupies the position F_1 where the force is in line with the centre O. Equilibrium is in this case stable, and F_2 is a position of unstable equilibrium analogous to the former one.

In the case of a sliding pair there is no position of equilibrium under such a force.

The case of a link in a mechanism is illustrated in Fig. 144. Here a force f acts on the link b, whose centrode is β, the small circle. The large circle δ is the centrode

for the fixed link *d*. Motion has to go on until the mechanism comes into a position in which the virtual radius of the point *F*, at present *FO*, will lie in the direction of the line *f*, that is, until the line *f* takes such a position as

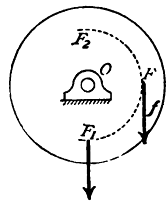


FIG. 143.

to cut the curve δ in the point which is the virtual centre for the time being. This position can be found in virtue of the condition that the virtual radius of a point is always at

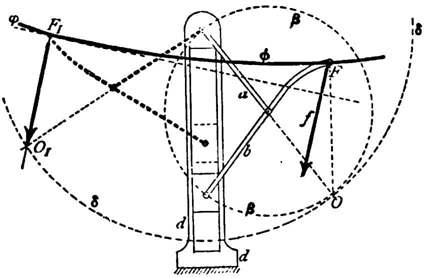


FIG. 144.

right angles to its path (§ 7). Let the path φ of the point *F* be drawn, and a tangent to it at right angles to the direction of *f*. Then the point of contact F_1 of this tangent is the position of *F*, and O_1 is the position of its

virtual centre, when the mechanism has attained its position of equilibrium, this position being shown in the dotted lines. It is in general out of the question to find the position of the tangent by any other means than "feeling" for it with a straight edge, so that in some cases the equilibrated position may not be very exactly obtained. The method given above is, however, quite general, no matter what the form of the mechanism may be.

There is another method which can be used here, and which can very often be applied in some form. The point O_1 must lie on δ, on f, and on a. If therefore the locus of intersections of f and a be drawn, its intersection with δ will give the point required.

The third case mentioned was that in which the force causing motion retained always the same position relatively to the moving body. In this case there is no position of equilibrium for one element either of a turning or a sliding pair, because if the force does not pass through the virtual centre to start with it can never do so. When the moving body is a link in a mechanism there may, or may not, be a position of equilibrium. In Fig. 145, the force f acting on the link c will bring it to rest in the dotted position, for f always cuts the centrode γ of the link c in the same point O_1, and therefore there must be equilibrium so soon as this point becomes the virtual centre. If the sense of f were the opposite to that of the arrow, motion would cease when O_2 became the virtual centre. But if the moving force were f_1 instead of f there would be no equilibrium, because there is no possible position in which the force f_1 could be acting through the virtual centre.

Our examples in this section have been chosen from mechanisms which have simple centrodes merely to make them more clear. The method is equally applicable in every

case, but for complex mechanisms or those with complex centrodes, the solution of the problem, although not different, is necessarily more troublesome.

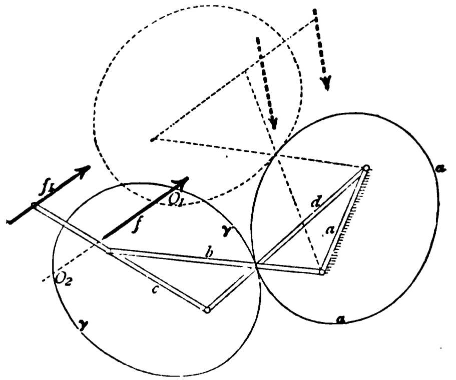


FIG. 145.

If different links of a mechanism are acted on by different forces there is in general no position of equilibrium, for it will only be in some very special case that the several forces come simultaneously into the condition necessary for equilibrium.

§ 43. FORCE AND WORK DIAGRAMS.

There are many cases in which it is important for practical purposes to know the conditions of equilibrium of a mechanism not only in one position, but in a series of consecutive positions, during, for instance, a whole revolution of a shaft. In this case it is often important to make a diagram showing

by its ordinates the successive values of the effort (p. 251) and the resistance which it balances. Such a diagram is entirely analogous to the velocity diagram of § 16, and one example will be sufficient to enable us to see its leading properties.

In Fig. 146 we have the familiar case of a steam engine mechanism. A variable driving effort acts on the cross-head, we require to find the variable resistance balanced by

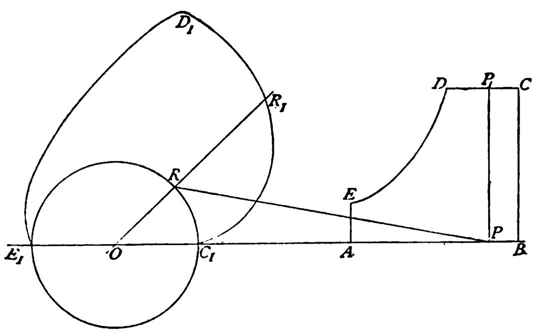


FIG. 146.

it at the crank pin. We know already that we may treat these forces (see § 40) as acting directly on the two ends of the connecting rod. The varying driving efforts (piston pressures) are of course given. It is convenient to set each effort up, as the ordinate of a curve, opposite the corresponding position of the piston, but at right angles to its actual direction. We thus get an *effort diagram* $ABCDE$, whose ordinate at any point is proportional to the pressure on the piston when it is opposite that point. The base of the diagram, AB, is the stroke of the engine. The diagram is in reality nothing more than the ordinates of the "indicator

card" of the engine set off to a straight base.[1] It is required to make some corresponding diagram of resistance. This may be done in two different ways. If PP_1 be the effort at any point P of the stroke, the corresponding crank pin resistance RR_1 can be found by the constructions of Figs. 115 or 121 of §§ 39 and 40 (the constructions are not repeated here). The resistance thus found can be set off radially either from the centre O or more conveniently from the crank pin circle as a base. By repeating this process for a sufficient number of points, a sufficient number of ordinates can be obtained to draw the curve $C_1D_1E_1$, whose ordinates, measured radially from the crank pin circle, represent the crank pin resistance for each position of the crank. The direction of the ordinates is here at right angles to the direction of the resistance, just as the direction of the ordinates of the effort curve CDE was at right angles to the direction of the piston pressures.

A curve like this is called a *polar diagram*, because its ordinates are measured radially from a centre or pole.

This diagram is the same as that which sometimes receives the name of "diagram of twisting moments." The crank pin resistance acts always at the same radius about O, and any diagram proportional to this resistance will therefore be proportional to the product of the resistance and its radius, which product is the moment tending to turn the shaft round, or to cause twisting in it—hence this name.

For many purposes it is more useful to make a diagram of resistance directly comparative with the diagram of effort. Such a diagram must have a straight base (see Fig. 147) whose

[1] It may be worth while to mention that in order correctly to treat an indicator card in this way its bottom line should be turned end for end before the pressures are measured, or, better, the bottom line of the card from the other end of the cylinder substituted for it.

length is the distance moved through by the crank pin during half a revolution, namely $AB \times \frac{\pi}{2}$. The curve $C_1D_1E_1$ in Fig. 147 shows the shape it would take in this case.

The ordinates of the curve CDE (Fig. 146) represent efforts, its abscissae (along AB), distances through which the efforts are executed. Its area therefore represents the product of effort and distance, or *work done by the piston.*[1]

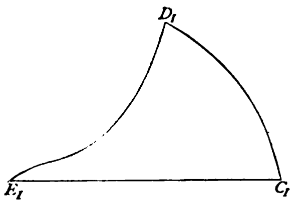


FIG. 147.

Similarly the ordinates of the curve $C_1D_1E_1$ in Fig. 147 represent resistances, and its abscissae distances through which they are overcome. The area under $C_1D_1E_1$ therefore also represents work—the *work taken up by the crank pin.* As we are here assuming the existence of static equilibrium in the mechanism, the work done by the piston must be exactly equal to that taken up by the crank pin, otherwise the latter would have either stored up or restored a certain amount of energy, and in either case its velocity would have changed. Hence the area of the effort diagram $BCDEA$, must be equal to that of the resistance diagram $C_1D_1E_1$, for each represents

[1] In general the ordinate represents pressure *per square inch of the piston area,* so that the area of the diagram would have to be multiplied by the number of square inches in the piston's area in order to find the whole work done.

the same amount of work upon the same scale. From this it follows that the mean height of the first diagram must be greater than that of the second in the inverse ratio of their bases, or in other words (as can, of course, be proved in numerous other ways) the mean crank pin resistance is less than the mean piston effort in the ratio $\frac{1}{\frac{\pi}{2}}$ or $\frac{1}{1 \cdot 57}$.

It is worth while pointing out that this reasoning is fatal to the very popular delusion that somehow or other work is "lost" in an ordinary steam engine because of the transformation of rectilinear into rotary motion, and especially because of the very small "leverage" at which the piston pressure acts when the piston is very near the beginning of its stroke. This notion has been the first cause of numberless rotary engine patents, and of other schemes, even more futile, for not letting the steam act on the piston until it had moved through some considerable part of its stroke, and so on. The mistake mainly arises from forgetfulness of the fact that when the piston is near the end of its stroke, and the piston pressure therefore acting with very small leverage to turn the crank, the velocity of the piston is correspondingly small in comparison with that of the crank pin. At every instant the relative velocities of crank pin and crosshead are inversely proportional to the forces at those points, so that during any time interval, long or short, the work done by one is exactly equal to the work taken up by the other, so long as the shaft is revolving uniformly.

The work done at a given point or on a given body by varying forces has often to be computed. In general it is obtained from a diagram such as that of Fig. 146, which is therefore called an *energy diagram* as often as a *force diagram*, its area measuring energy while its ordinates measure force. The most familiar case of such a diagram is the

indicator card of a steam engine already mentioned, but as generally worked out the indicator card is rather a force- than a work-diagram. To obtain *work* from it, only its mean effort or pressure is generally found by measurement, and this is multiplied by the area of the cylinder (see foot-note on p. 323) and by the distance through which the effort is exerted in a minute, *i.e.*, by twice the stroke × the number of revolutions per minute. This gives the work in foot-pounds actually done per minute, and is usually reduced to "horse power" (p. 254) by dividing by 33,000. But the diagram may be used somewhat differently. If for instance the mean effort were simply multiplied by the length of the diagram AB (measured on the distance scale), and by the area of the cylinder, the product would be the work done per (single) stroke, in foot-pounds. Or the mean effort need not be directly measured at all, but the area of the diagram determined by a planimeter or otherwise, and converted at once into foot-pounds of work by a proper multiplier, which would include in itself the area of the cylinder. Or lastly the length AB may be taken on a scale of *volumes* instead of distances, and the pressures or efforts taken on corresponding units of area. Thus if AB be set off on a scale of cubic feet, and the ordinates represent pressures *per square foot*, the area of the diagram gives foot-pounds of work done in the cylinder at once, without multiplication by any constant. More commonly the pressures are still taken per square *inch*, so that the area would have to be multiplied by 144 to convert it into foot-pounds.

In setting off the indicator cards of compound engines it is usually this latter method which is most convenient (making AB represent volumes); in other cases it is generally most handy to use one of the former methods, with abscissae along AB measured as distances.

CHAPTER IX.

PROBLEMS IN MACHINE DYNAMICS.

§ 44.—TRAIN RESISTANCE.

In the commencement of the last chapter it was pointed out that as long as a body was not actually changing its form it was said to be *in equilibrium.*

Further, we have seen that although any force, however small, inevitably alters the form of the bodies, however rigid, upon which it acts, yet that these alterations of form are, in a machine, intentionally made so minute that we are able to neglect them, and to treat the various bodies forming part of the machine as if their forms really remained unchanged.[1] We called the conditions of equilibrium **static** in the case where there were acting only external forces and pressures [2] in addition to the stresses in the links. This, we saw, corresponded to a condition either of rest or uniform velocity on the part of each body in the machine. We have examined in the last chapter most of the principal problems connected with the static equilibrium of bodies having constrained motion. In the present chapter we shall examine

[1] We neglect here the case of springs, leather belts, and the one or two other instances in which the alteration of form under external force is comparatively great.

[2] See § 36, p. 263.

a number of problems of a kind often distinguished from the former as dynamic instead of static, but which it is perhaps better to distinguish as problems of **kinetic**, instead of **static, equilibrium.** The difference between the two conditions is not the difference between no force and force, or between rest and motion, but between rest *or* motion with uniform velocity, and motion with varying velocity, that is, with acceleration. In this case not only external forces and pressures act upon the links, but also resistances due to their acceleration, which resistances may be positive or negative according to the sense of the acceleration. These resistances are forces whose magnitudes depend upon the *masses* of the links themselves, and are in this respect essentially different from the forces and pressures previously considered, in working with which the masses of the bodies acted upon never entered into the question. It is on this account that we separate the problems in which they occur from those formerly considered, and not because both sets of problems do not alike relate to conditions of equilibrium.

In the present chapter we shall examine in detail certain typical and very important problems of kinetic equilibrium representing those which have actually to be solved in engineering work. We shall take first in this section, as perhaps the simplest case that can be taken, the case of the motion of a train, from which we have already used several illustrations in § 28. We shall next examine dynamically a simple direct-acting pumping engine, then a Cornish engine, then the driving mechanism of an ordinary horizontal steam engine, next the fly-wheel of such an engine, and its connecting rod, and lastly the case of centrifugal governors.

Let us suppose that a train weighing 80 tons, or say 180,000 pounds, is running on a level at 40 miles per hour,

and that 40 wheels are simultaneously braked, with such a pressure on each brake as gives a frictional resistance of 30 pounds at the periphery of each wheel. The total brake pressure is thus $40 \times 30 = 1200$ pounds. The speed of the train is 58·4 feet per second. It is required to draw a diagram of the stop of the train.

It may be stated at once that in such a very simple case as this one, diagrams are by no means necessary, nor is a graphic solution to be preferred to ordinary calculation if only final results are required. Both calculation and diagram will be given here, the latter partly for the sake of the determination of its scales, and partly because by its nature it gives not only a final result—the distance run before stopping, or whatever it may be—but also a pictorial representation of the whole process of stopping, which it is in many cases important to follow, and which otherwise can only be understood by a series of separate calculations, together much more trouble than the drawing of the diagram.

The distance which will be run before stopping can be found at once by calculating the kinetic energy stored up in the train, as it moves with the given velocity, and dividing this quantity by the resistance to its forward motion, namely, the resistance of the train proper plus the added artificial resistance of the brakes. The stored-up energy is

$$\frac{180{,}000 \times 58{\cdot}4^2}{64{\cdot}4} = 9{,}530{,}000 \text{ foot-pounds.}$$

The normal resistance of the train on a level may be taken as 8 pounds per ton, or 640 pounds. The brake resistance is 1,200 pounds, but it has to be overcome through a distance π times as great as that moved through by the train as a whole. The total resistance is therefore $640 + (\pi \times 1200)$ = (say) 4,400 pounds. The distance which the train will

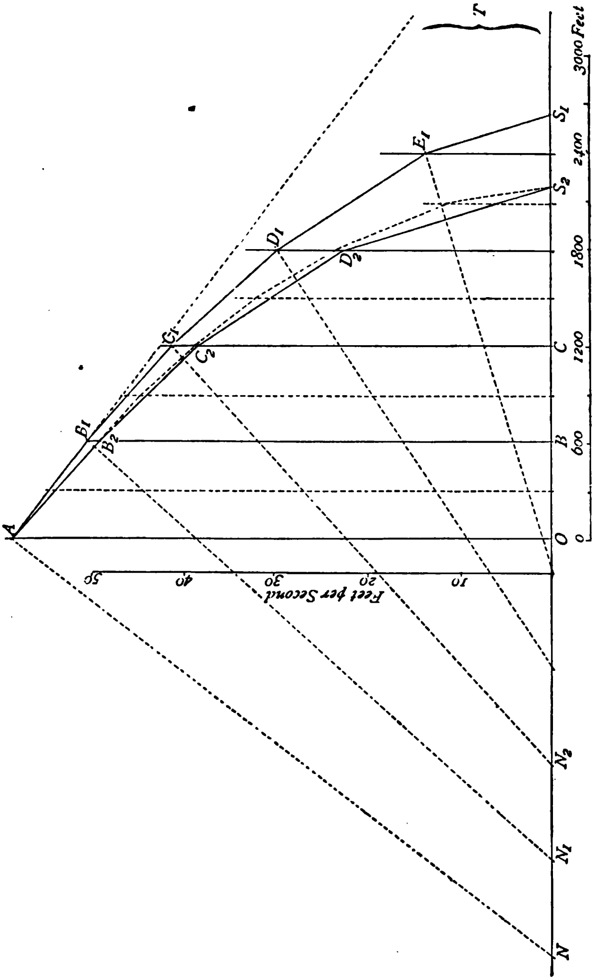
run before stopping is therefore $\frac{9,530,000}{4,400} = 2,170$ feet, or about two-fifths of a mile.

The diagram (Fig. 148) which represents this case is drawn with the following scales :—

(1) Force or pressure scale 2,000 pounds = 1 inch.
(2) Distance scale 1,200 feet = 1 inch.
(3) Velocity scale 20·6 feet per second = 1 inch.
(4) Acceleration scale 0·356 foot-seconds per second = 1 inch.

The first two scales are taken arbitrarily as may be convenient. The acceleration scale is derived from the force scale by the relation $a = f\frac{g}{w}$ (p. 222), which tells us that 1 inch must stand for $\frac{g}{w}$ times as many units of acceleration as of pressure. This fraction is here ·000178, so that the acceleration scale is 2000 times this, as given above. The velocity scale is derived from the distance and acceleration scales also as before, one inch standing for $\sqrt{0·356 \times 1200}$, or 20·6 feet per second (very nearly).

The distance OA is first set up for the initial velocity of the train, and ON for the acceleration (here negative, of course). AB_1 is drawn at right angles to NA. BB_1 is a vertical at any convenient distance from the origin. BN_1 is the acceleration (= ON), and the next segment of the velocity curve, B_1C_1, is drawn at right angles to N_1B_1 and so on. This gives the curve $AB_1C_1 \ldots S_1$, and the distance OS_1 run before the train stops (see p. 204). But the process is obviously one which gives cumulative errors, and these are too great to be neglected. For not only is the point C_1, for instance, too high in position on account of the substi-



A
B_1
B_2
C_1
C_2
D_1
D_2
E_1
S_1
S_2
O
B
C
N
N_1
N_2
T
0
600
1200
1800
2400
3000 Feet
10
20
30
40
50
Feet per Second

FIG. 148.

tution of the straight line B_1C_1 for an arc convex upwards, but it is further misplaced by the similar error in the position of B_1, which makes the angle BB_1C_1 too great. The error of each point in the curve is thus greater than that of the preceding one, and the whole distance OS_1 is (in this example) nearly 20 per cent. too great. The longer the distance intervals OB, BC, &c. be taken the greater does this error become. For any reasonable distance intervals, however, the point S_1 can be obtained very approximately by drawing a second polygon $AB_2C_2 \ldots S_2$, so that AB_2 is parallel to B_1C_1, B_2C_2 parallel to C_1D_1, C_2D_2 to D_1E_1 and so on. This construction makes the distance OS_2 in the figure very slightly too small. But although this error of S_2 is here quite negligible, yet with the distance intervals shown the intermediate points B_2C_2 and D_2 are very sensibly out of position. By taking the distance intervals sufficiently small, these points also can be brought sensibly right. In Fig. 148 the dotted curve is drawn like $AB_2C_2S_2$, but for distance intervals $= \frac{OB}{2}$ or 300 feet, and its points sensibly coincide with calculated points.

As the acceleration, which is the sub-normal to the velocity curve (p. 205), is constant, we know that the curve itself is a parabola, and as such it can easily be drawn. The point S_2 must lie midway between O and T, where AB_1 cuts the axis. In starting such a diagram the two quantities to be set out are the velocity and the acceleration. It is not really necessary, however, to set out the latter, for we know that it is proportional to the force, hence ON is really set out equal to the force (here resistance) causing (negative) acceleration. Knowledge of the scale on which ON represents the acceleration is only required in order to find on what scale the velocity must be drawn in order to correspond to our assumed

distance scale, or—if the velocity scale has been arbitrarily assumed—to determine on what scale OD must be read of as distance.

Fig. 149 shows the same problem worked out on a time, instead of a distance, base. The velocity curve is here, as

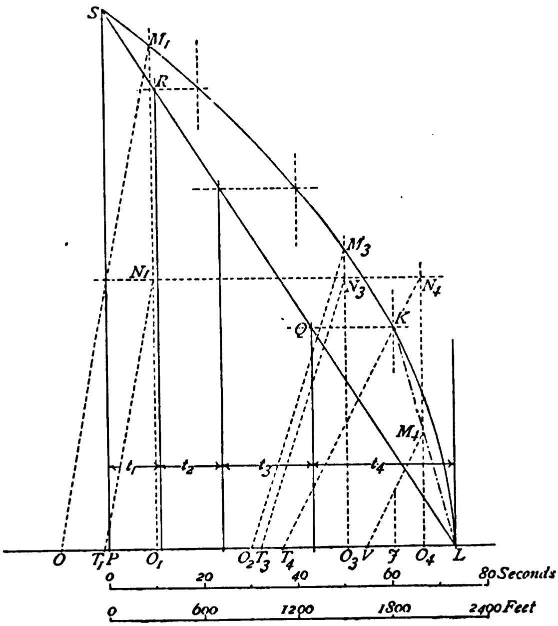


FIG. 149.

in Fig. 92, a straight line, the acceleration being constant. The acceleration is equal to $\frac{f}{m}$ or $f\frac{g}{w}$, which is here

$4400 \times \dfrac{32 \cdot 2}{180{,}000} = 0 \cdot 787$ foot-seconds per second. The

time of the stop (we have already measured its *distance*) is $\frac{58\cdot4}{0\cdot787} = 74\cdot2$ seconds, or about a minute and a quarter. The time scale in the figure is taken so as to make the total length of the base the same as in the former case (see § 28 p. 211). This necessarily makes the actual time scale an odd one. As the diagram is drawn $\frac{2,170}{1,200}$ inches stand for 74·2 seconds, so that the time scale is 41 seconds per inch.

A part of the construction, which is the same as that of Fig. 100, described on p. 209, is given in Fig. 149. *SKL* is simply a copy of the velocity curve of Fig. 148. O_1O is a length equal to one distance interval, set back from O_1, the centre of the first interval. O_1N_1 is a distance determined by the method of p. 209. In this case there are $\frac{20\cdot6}{1200} = 0\cdot0172$ as many units of velocity as of distance in one inch, so that $n = 0\cdot0172$. The distance O_1N_1 must therefore be made equal to $\frac{1}{0\cdot0172} = 58\cdot1$ time units. The time unit being (as found above) $\frac{1}{41} = 0\cdot0244$ inch, the length O_1N_1 must be $0\cdot0244 \times 58\cdot1 = 1\cdot42$ inches. N_1T_1 being drawn parallel to M_1O, the distance $O_1T_1 = t_1$ represents the time taken by the train in traversing the first distance interval, and R is therefore a point on the new velocity curve. Similarly $O_3T_3 = t_3$ represents the time taken in traversing the third distance interval, and Q is a point on the new velocity curve. This curve as a whole is simply the straight line *SRQL*, as in Fig. 92. It must not be forgotten that in getting the time $O_4T_4 = t_4$ for the last distance interval, the distance O_4V must be made equal to

JL, the actual length of that interval and set back from its mid point. It is better to take M_4 on the chord than on the arc *KL*.

In all these cases, as has been already pointed out, the diagram has been scarcely more than illustrative of the problem—the actual answer has in each case been worked out quite independently of it. With altered data, however, such as are quite likely to occur in practice, it is often convenient to use the graphic method for its own sake, although even then its comparative convenience is not nearly so great as in the other cases which we have to consider in this chapter.

Let us now suppose that the stop is not to be made on a level line, but on one which has first an uphill gradient of 1 : 250 for 1,000 feet, is then level for 500 feet, then has a down-hill gradient of 1 : 150. On the first section the resistance will be $\frac{180{,}000}{250} = 720$ pounds *more* than previously, or 5,120 pounds in all; on the second section everything will be precisely as before; on the last section the resistance will be $\frac{180{,}000}{150} = 1{,}200$ pounds *less* than before, or 3,200 pounds in all. These quantities would be used in the diagram as the sub-normals in the first, second, and third sections respectively. At the top of the hill the velocity of the train has been reduced to 39·7 feet per second. At the end of the level ground its velocity is 28·0 feet per second, while the energy still stored up in the moving train is 2,210,000 foot-pounds. With the now diminished negative acceleration due to the downward gradient, the train will still not come to rest for 690 feet, so that the whole stop will occupy 2,190 feet. As the (negative) acceleration is constant on each of the three sections, the velocity curve consists of arcs of three parabolas, all having the same axis. As the

distance run in each case before stopping varies directly as the resistance and as *the square* of the initial velocity, it is especially important that the train should enter any section where the resistance is diminished (such as the down-hill section in the case supposed) with as small a velocity as possible.

We may take one more example in this section. Let it be supposed that a train of the same weight, &c., as before, is two miles from a station and running up a continuous incline of 1 in 250, in the middle of which the station is placed. How long will it take to reach the station, and how far would it over-run the station if the brakes "leaked off" suddenly after being on for 20 seconds? To answer the first question we have first to find when the brakes have to be applied in order that the train may stop just at the station. The total resistance of the braked train on an upward gradient of 1 in 250 is 5,120 pounds. The stoppage from a speed of 40 miles per hour under this resistance will require $\frac{9,530,000}{5,120} =$ 1,860 feet. Before applying the brakes the train will therefore have run 8,700 feet at 40 miles an hour, which will occupy 149 seconds. The negative acceleration, when the brakes are applied, will be $5,120 \times \frac{32 \cdot 2}{180,000} = 0 \cdot 916$ foot-seconds per second, and the duration of the stop itself therefore $\frac{58 \cdot 4}{0 \cdot 916} = 64$ seconds. The whole time taken up in running the two miles will be 213 seconds.

If the brakes leaked off suddenly at the end of 20 seconds the train would be left running, under its own proper resistance only, at a speed reduced by $0 \cdot 916 \times 20 = 18 \cdot 3$ feet per second. The speed of the train is therefore (say) 40 feet per second, the kinetic energy at this speed is 4,480,000

foot-pounds, and the resistance of the train, without the brakes, $640 + 720 = 1,360$ pounds. The train will therefore run on $\frac{4,480,000}{1,360} = 3,290$ feet before it stops. When the brakes were applied the train was 1,860 feet from the station. In 20 seconds this distance must have been diminished by 984 feet, leaving the train 876 feet from the station, which it would therefore over-run by 2,414 feet, or nearly half a mile.

From these data there can at once be calculated the velocity with which the train would strike any obstacle which happened to be standing in its way at the station.

§ 45.—DIRECT ACTING PUMPING ENGINE.

In one of the simplest forms of pumping engines, known as the "Bull Engine," the cylinder is placed vertically above the pump shaft, and the pump rods or "pitwork" simply hung direct to the piston rod. Kinematically the combination is nothing but a sliding pair of elements, there being no crank or rotating parts of any kind. Dynamically the machine is of much more interest, and its action much more complex. For although the form of the piston and cylinder prevent any relative motions but those of an ordinary sliding pair, yet they do not in any way affect or control the velocity of motion, and the length of stroke of the engine is entirely dependent on the accelerating forces in action, and it both may and does vary from stroke to stroke instead of being an absolutely fixed distance as in an ordinary engine. Buffer blocks are provided for the crosshead to strike against at each end, but if the accelerating forces exceed certain limits these may be destroyed and the cylinder cover or end knocked out.

We shall investigate the condition of the working of an engine of this type by the aid of the principles and constructions already examined.

Let there be given, as in Fig. 150, a diagram $AA_1B_1C_1$ etc., whose ordinates represent the steam-pressures below the piston tending to lift it. The point at which this curve ends on the right we do not know, because the stroke of the engine is not a fixed quantity, but one which we have to find out. Further let there be given the weight of the whole pitwork, AA_0, which forms the resistance against which the piston has to rise, and which is constant throughout the stroke, whatever its length may be. The horizontal line $A_0 \ldots . F_0$ will then form a resistance diagram. From these data our first problem will be to find the velocity at different points of the stroke, *i.e.*, to construct a curve whose ordinates shall measure these velocities.

The actual effort available for producing acceleration is at each instant the difference between the total effort and the resistance, as A_0A_1, B_0B_1, E_0E_1 etc. During the first part of the stroke these efforts are positive, and the speed of the piston will therefore increase. During the second part they are negative—the resistance being greater than the effort—and the acceleration is therefore also negative, the speed of the piston diminishing until at length it becomes zero, and the piston stops. The total resistance AA_0 is partly due to the gravity of the mass and is partly frictional, but in this case we may neglect this latter part, and treat the ordinate AA_0 as representing simply the weight of the pitwork. If we take the ordinate AA_1, (as is commonly done) to represent not the *whole* pressure on the piston, but the pressure *per square inch*, then AA_0 must represent also the resistance *per square inch*, *i.e.* the total resistance divided by the area of the piston. There is therefore a nett effort A_0A_1

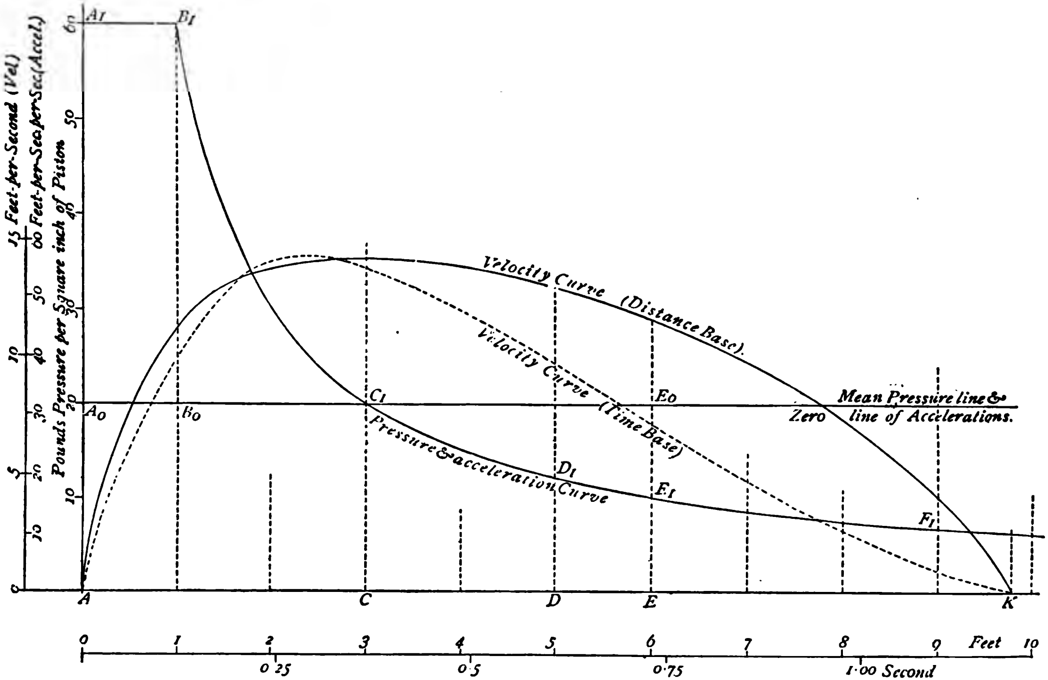

15 Feet-per-Second (Vel.)
60 Feet-per-Sec-per-Sec (Accel.)
Pounds Pressure per Square inch of Piston
A1
B1
A0
B0
C1
D1
E0
E1
F1
A
C
D
E
K
Velocity Curve (Distance Base).
Velocity Curve (Time Base)
Pressure & acceleration Curve
Mean Pressure line & line of Accelerations.
Zero
Feet
0.25
0.5
0.75
1.00 Second

FIG. 150.

per square inch of piston to be expended in accelerating a mass equal to AA_0, also per square inch of piston. This mass is constant throughout, so that the acceleration of the pitwork at each instant is simply proportional to the force producing it, that is, to the ordinate between the line $A_0 \ldots F_0$ and the curve $A_1B_1 \ldots F_1$, and is positive or negative according as that curve lies above or below the line $A_0 \ldots F_0$. The curve $A_1B_1 \ldots F_1$, may therefore be taken simply as an acceleration curve, *the scale of which is as yet unknown*, on a distance base, and the problem is the one treated already in Fig. 99, given an acceleration diagram on a distance base, to construct from it a curve of velocities. The construction, being exactly the same as that of Fig. 99, is not repeated here,—the velocity curve is shown in the figure. The velocity reaches a maximum where the acceleration is zero (at C_1) and then diminishes under the negative acceleration until at K the curve cuts its axis. At this point therefore, motion ceases, and the stroke is completed, its length being AK, or 9·8 feet on the scale used.

If there has been room for this stroke in the cylinder, the piston will simply have come to rest clear of the cylinder cover by a certain distance. But our construction has obviously been quite independent of any particular length of cylinder. The piston would naturally come to rest at K, no matter how long the cylinder was, and if the space in the cylinder available for its motion were *less* than AK,—say AD,—the piston would strike the cylinder end when it had travelled so far, and the cylinder end would be broken unless it were strong enough to stand the blow. As such an accident would be very serious in an engine, it is guarded against, as mentioned above, by placing buffers or buffer beams of some description outside the cylinder in such a position that the crosshead must strike them before the

piston strikes the end of the cylinder. In spite of these precautions, however, accidents of the kind mentioned have not unfrequently occurred.

The velocity diagram has now been constructed, and the length of the stroke formed, but we do not yet know on what scale we can measure the velocities, for the scale of the acceleration diagram has not been determined. This can be done easily from the general relation between force and acceleration, viz., $a = f\frac{g}{w}$, which tells us that if $f = 1$, $a = \frac{g}{w}$, so that the length which stands for one unit force (here 1 pound per square inch of cylinder area) stands for $\frac{g}{w}$ units of acceleration. As w is here 20, $\frac{g}{w} = 1 \cdot 6$; the force scale of the figure as drawn was 20 to the inch, hence the acceleration scale is $(20 \times 1 \cdot 6)$ or 32 foot-seconds per second to the inch.

In section 28, p. 207, we have already seen how to find the velocity scale when distance and acceleration scales were given. The distance scale of the figure was originally drawn 2 feet to the inch, there are therefore $\frac{32}{2} = 16$ units of acceleration per unit of distance on the paper, and there must be $\sqrt{16}$ or 4 units of velocity per unit of distance. The velocity scale would therefore be 8 feet per second to the inch, and by measurement the maximum velocity would be $14 \cdot 1$ feet per second.

There is another way in which the maximum velocity, or the velocity at any point of the stroke, can be found, but it is less convenient than the foregoing unless the object is

merely to find the maximum velocity.[1] From § 33 we know that there is stored up in the moving masses, a quantity of energy $E = m\frac{v^2}{2} = \frac{wv^2}{2g}$. This energy E we saw to be equal to the product of the mean force causing the acceleration into the distance through which the body had been caused to move during the operation. In our diagram this is simply equal to the area of $A_1B_1C_1A_0$; for the length A_0C_1 is equal to the distance just mentioned and the ordinates of $A_1B_1C_1$ measure the forces causing acceleration. By construction or calculation the area of $A_1B_1C_1A_0$ will be found to be equal to 60·8 foot-pounds (per inch of piston area), while w, we have seen, is equal to twenty pounds (also per square inch of piston area). From this $v = \sqrt{\frac{2Eg}{w}} = \sqrt{\frac{2 \times 60.8 \times 32}{20}} = 14{\cdot}0$ feet per second, which checks well with the 14·1 feet per second found by the other method.

To find the velocity at any other point, as B_0, the area $A_1B_1B_0A_0$ would have to be taken intead of that used above. If E_0 were the point where the velocity was required, we should have to take for the stored-up energy the value of the area $A_1B_1C_1D_1E_1E_0A_0$, of which a part $C_1E_1E_0$ is *negative*, and would have to be subtracted from the rest, the stored-up energy of course diminishing as it expends itself in keeping the body moving after the effort falls short of the resistance. But if the velocities at a number of points are wanted, these calculations become tedious, and it is simpler to draw the velocity curve at once.

The finding of the duration of the stroke is not a very

[1] The graphic treatment of this method was given at length by the author in two papers in *Engineering*, vol. xx., pp. 371, 409.

difficult matter. It is only necessary to reduce the velocity curve to a time base by the method of p. 209, and find the length of the base. The mean velocity of the piston is found of course by dividing the whole stroke by the time occupied by it. It is in this case 8·17 feet per second. The time taken to traverse the whole stroke is 1·2 seconds. The velocity curve on a time base equal in length to the distance base (as in the last section) is shown in the figure. The time scale is 4·1 inches per second.

An engine such as we are now discussing does nothing on its up stroke but lift the weight of the pitwork, or such of that weight as is not balanced.[1] The whole of the pumping is done on the down stroke. The two ends of the cylinder are put into free communication, so that the piston is in equilibrium so far as the steam is concerned. The weight of the pitwork is made somewhat greater than that of the column of water in the rising main, so that the pitwork descends, carrying the piston, etc. with it, and the water is forced upwards. Again there is no kinematic means of limiting the stroke, the length of which depends entirely on the various forces in action, and we may now proceed to examine these.

The weight of the pitwork is equivalent to twenty pounds per square inch of cylinder area. The water load may be taken as sixteen pounds per square inch, and the valve and other resistances as three pounds per square inch. There is therefore a nett force of $20 - (16 + 3) = 1$ pound per square inch available to cause acceleration of the whole mass $(20 + 16)$ or 36 lbs. per square inch. The valve resistances of 3 pounds per square inch do not represent, of course, any mass to be

[1] The case where a considerable part of the weight is balanced, so that the whole mass set in motion is much greater than the mass of the weight lifted, is examined in § 46.

accelerated, and therefore are not here taken into account. The designer of the engine determines beforehand the point up to which these shall be the only forces acting. Let us take this point as 7·9 feet from the beginning of the return stroke, or 2 feet short of the point at which the piston originally commenced its motion. During this distance the acceleration will be constant, and will be

$$a = f\frac{g}{w} = 1 \times \frac{32'}{36} = 0{\cdot}9$$

nearly. The velocity is

$$v = \sqrt{2as} = \sqrt{2 \times 0{\cdot}9 \times 7{\cdot}9} = 3{\cdot}8$$

feet per second nearly. This is the velocity of the mass starting from 0, after it has traversed a distance of 7·9 feet under a uniform acceleration of 0·9 foot-seconds per second.

In order to bring the moving mass to rest within the short distance that remains, it is necessary to interpose a resistance very much greater than the force (one pound per square inch) which has been accelerating the mass hitherto. This is done by shutting off the communication between the two ends of the cylinder so as to leave a "cushion" of steam imprisoned below the piston. The downward motion of the piston therefore continues against a gradually increasing resistance due to the compressed cushion of steam, the pressure above it gradually diminishing, at the same time, as the steam expands. In Fig. 151 the curve *m m* represents the increasing pressure below the piston, *n n* the decreasing pressure above it. The ordinates between *m* and *n* therefore represent the resistance caused by the cushioning. During the whole process the original accelerating force of 1 lb. per square inch continues to act, so that to find the nett resistance, or force causing the negative acceleration under which the piston is brought to rest, this must be subtracted, leaving the nett resistance equal to the ordinate

between the curves *p p* and *m m*. These ordinates, transferred and set off from the axis, form the curve *r r*. The piston must continue to increase in velocity, although at a slower rate, until the point *R* is reached, where the interposed resistance is just equal to the original effort. After that point its velocity must continually diminish until it comes to rest. The curve *r r* can be treated as an acceleration curve in the same way as we treated the steam curve $A_1B_1C_1$, but not measured on the same scale. For while here the length that stands for unit force still stands for $\frac{g}{w}a$, the weight *w* is no

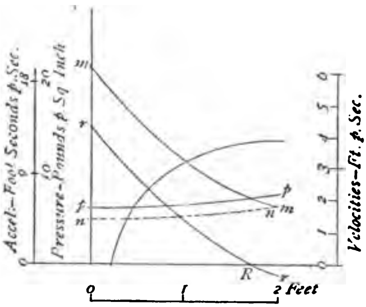


FIG. 151.

longer what it was, 20 lbs. per square inch, but is 36 lbs. per square inch. $\frac{g}{w}$ is therefore $\frac{32}{36}$, or 0·9, and if the pressure scale is 20 pounds to the inch (as originally drawn) the acceleration scale will be $(20 \times 0{\cdot}9) = 18$ foot-seconds per second to the inch. The distance scale being (as before) 2 feet to the inch, the ratio $\frac{18}{2} = n^2 = 9$, and the velocity scale is $(2 \times \sqrt{9})$ or 6 feet-per-second to the inch. On this

scale the calculated velocity, 3·8 feet per second, is set off as v_1 at the instant when cushioning commences, and the rest of the velocity diagram drawn as before. With pressures and proportions such as we have chosen the piston will come to rest about three inches short of the point at which it started. In actual practice consecutive strokes of a Cornish pumping engine may often vary as much as this. It should, perhaps, be mentioned that the final pressure KK_1 in Fig. 150 (and therefore the corresponding pressure in Fig. 151) is drawn considerably lower than it would probably be in an engine of the type described.

§ 46.—CORNISH PUMPING ENGINE.

The working out of a Cornish Beam Engine as we worked out the Bull Engine in the last section is a little more complex, but not different in principle. In the "indoor" or down stroke, where the work done is simply the lifting of the pitwork, there is now, besides the mass of the pump-rods, the whole mass of the engine beam to be accelerated. This mass, being balanced, adds nothing to the resistance (except so far as the friction in its journals is concerned) but simply "dilutes" (see p. 225) the acceleration. It will be sufficient here to give some data of a case and to work out the scales as examples. Let the total resistance on the indoor or down stroke be 16 pounds per square inch of piston, made up of 14 pounds pitwork and 2 pounds per square inch frictional resistances. Further, take the beam's weight as equivalent to 4 pounds per square inch of piston at the radius of its end points. The pressure scale is, say, 12 pounds (per square

inch of piston) to the inch. The mass moved is $(14 + 4) = 18$ pounds per square inch. As $\frac{g}{w}$ is here $= \frac{16}{9}$, the length which stands for unit force stands for $\frac{16}{9}$ units of acceleration. The acceleration scale is therefore $12 \times \frac{16}{9} = 21{\cdot}3$ units per inch. If the distance scale is 2 inches to the foot, *i.e.* half a unit per inch, the value of n^2 is $\frac{21{\cdot}3}{0{\cdot}5} = 42{\cdot}6$, and $n = \sqrt{42{\cdot}6} = 6{\cdot}5$ (nearly). The velocity scale would therefore be $(0.5 \times 6{\cdot}5) = 3.25$ feet-per-second to the inch.

It is unnecessary to work out this example also for the "outdoor" or pumping stroke, as its conditions are so similar to those of the last case.

It is well to mention that very often the mass to be accelerated is very much greater than that represented by the beam, &c., and the unbalanced part of the pitwork. In such a case as that supposed above, for example, the total weight of the pitwork might be 30 pounds per square inch of cylinder, but 16 pounds of this might be balanced by counter-weights. Then the actual weight to be lifted would remain as only 14 pounds per square inch of piston, but the weight of the mass to be accelerated would be $14 + (16 \times 2) = 46$ pounds per square inch of piston. Then $\frac{g}{w}$ would be $\frac{32}{46} \doteq 0{\cdot}695$, and the length which stands for unit force would stand for $0{\cdot}695$ units of acceleration. The acceleration scale would be $12 \times {\cdot}695 = 8{\cdot}34$ units per inch instead of $21{\cdot}3$ as above. If the distance scale were (as assumed

above) 2 inches to the foot, the value of n^2 would be $\frac{8 \cdot 34}{0 \cdot 5}$ $= 16 \cdot 7$ and $n = \sqrt{16 \cdot 7} = 4 \cdot 1$ nearly. The velocity scale would therefore be 2·05 feet-per-second to the inch.

§ 47.—ORDINARY STEAM ENGINE.

The dynamics of the mechanism forming the driving train of an ordinary steam engine is simpler than the dynamics of the Cornish engine, although the mechanism itself is so much more complex. The principal cause of this difference is that in the ordinary steam engine special means are used to keep the velocity of one point [1] (the crank pin centre) as nearly constant as possible, and from this velocity to control and determine at every instant the velocity of all other points. Not only does the connection of the piston rod with the crank make the stroke an absolutely fixed quantity, but small variations in the driving pressure on the piston are sensibly without effect on its velocity. Even supposing the steam to be entirely shut off, the energy stored up in the fly-wheel of the engine would be sufficient to keep it running for a considerable number of revolutions before coming to rest. So long as the engine is working in normal fashion, therefore, we may consider that the velocity of the crank pin is uniform, and that it forms one of the data of our problems, and may determine from it, by the methods of § 16, the velocities at any instant of any other points in the mechanism. This assumption that the crank pin undergoes no accelera-

[1] Or we may say, if preferred, the angular velocity of one link, namely, that formed by the crank, crank pin, and crank shaft.

tion involves the assumption that the mean driving effort exactly balances the mean resistance, so that whatever small quickenings and slackenings of speed there may be within small fractions of a second, on the whole there is neither the one nor the other. Our assumptions are fully justified in practice;—whatever governing arrangement is used to keep the speed of the shaft unchanged does so simply by automatically varying the effort as the resistance varies, and so keeping up the balance between them.

The principal practical problem connected with the dynamics of the steam engine is usually:—Given the velocity at each instant of the "reciprocating parts," to find their accelerations, and to find how much effort is absorbed by the masses, or given back by them, and at what times, in consequence of the given changes of velocity. The only strictly "reciprocating parts" are the piston and piston rod, cross-head and guide blocks (if any). It is not usual, however, to treat the connecting rod separately, so that half its weight is generally assumed to be added to that of the parts just mentioned and to move with them, an assumption which does not involve any considerable error, as is shown in § 49.

In the form in which we have been treating these problems this means:—Given a velocity curve on a distance base for certain masses, to draw the acceleration curve, find its scale, and also its scale as a curve of resistance. The converse case to this we treated in §§ 44 and 45, where we used a force or pressure curve as a curve of acceleration and found the scale on which we could read it as such. We shall for the present neglect the obliquity of the connecting rod, that is, assume the rod to be infinitely long. We shall also simplify matters, as we did before, by taking all weights, etc., *per square inch of piston area* unless otherwise distinctly stated.

Under these conditions, if we take a length equal to the radius of the crank pin to represent its (uniform) velocity, the velocity diagram for the cross-head will be a semicircle of the same radius (compare Fig. 53, p. 109). The mean velocity of the crank pin and the maximum velocity of the reciprocating parts will in this case be equal, viz. $2\pi r t$ feet per second, where r is the radius of the crank in feet, and t the number of turns which it makes in a second. As the same length on the paper stands for r feet on the distance scale and for $2\pi r t$ feet-per-second on the velocity scale, there are $(2\pi t)$ times as many units of velocity as of distance in a given length. Hence if the distance scale be d feet per inch the velocity scale is $(2\pi t)\ d$ feet-per-second to the inch, and the acceleration scale (see p. 207) will be $(2\pi t)^2 d = 4\pi^2 t^2 d = 39{\cdot}4\ t^2 d$ foot-seconds per second to the inch.

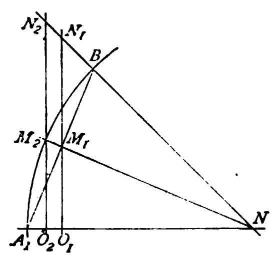


FIG. 152.

All normals to the velocity curve, as M_2N in Fig. 152, as well as all central normals to chords of that curve (as M_1N) pass through the centre N. Setting up the acceleration O_1N through M_1 or O_2N through M_2 will equally give points N_1 or N_2 on a line passing through the centre N and making

an angle of 45° with the axis, for NO_1N_1, NO_2N_2, etc., must each be a right-angled triangle with the sides containing the right angle equal. The acceleration curve is therefore a straight line making an angle of 45° with the axis and cutting it in the centre N. The maximum ordinates of this line will be at the ends of the semicircle (that is, at the beginning and end of the stroke) and will be equal in length to the radius AN or r.

As the mass accelerated is constant throughout the stroke, the ordinates of the acceleration curve must be proportional to the accelerating forces or resistances. To read the acceleration curve as a force diagram we have therefore only to remember that the length which stands for unit acceleration stands for $\frac{w}{g}$ units of force, so that there are $\frac{w}{g}$ times as many units of force per inch as there are units of acceleration, w being the weight of the reciprocating parts in pounds per square inch of cylinder area.

The force or pressure scale is therefore one inch on paper to $\left(39{\cdot}4 \quad t^2d.\ \frac{w}{g}\right)$ pounds per square inch. Any such scale would of course be excessively inconvenient in practice, but it is fortunately unnecessary to use it. We can find by its help the value of the maximum accelerating force in terms of the revolutions, the weight or mass of the reciprocating parts, and the radius of the crank pin. This can then be set off at the commencement and end of the stroke on whatever scale it has been found convenient to use for the pressures, and the acceleration line at once drawn to that scale. The length which stands for the maximum acceleration is equal to r feet on the distance scale, or actually $\frac{r}{d}$ inches on

the paper. The force or pressure per square inch of piston area required for the acceleration is therefore

$$39{\cdot}4\, t^2\, d.\, \frac{w}{g} \times \frac{r}{d} \text{ pounds}$$

$$= 1{\cdot}224\, t^2 w\, r \text{ pounds,}$$

or if we take T for the revolutions per *minute*

$$= {\cdot}00034\, T^2 w\, r \text{ pounds.}$$

At the beginning of each stroke this pressure has to be *deducted* from the pressure shown by the indicator cards to find the nett forward pressure, at the end of the stroke it has to be *added to* the indicated pressure. At mid-stroke, where there is no acceleration, the indicated pressure of course remains unaltered. The formula shows that the resistance due to acceleration varies as the square of the speed or number of revolutions per unit time, and directly as the weight of the reciprocating parts and as the crank-pin radius. It will be noted that the value of this resistance at each end of the stroke, where the velocity of the parts is zero, is just the same as the "centrifugal force" (see p. 228) which they would have if they were revolving with the crank pin for that instant.

With old-fashioned slow-speed engines this resistance was small enough to be neglected without considerable error. In modern quick-running engines it often bears a very large ratio to the indicated initial pressure, and cannot be neglected without serious risk of impairing the efficiency of the engine, as will be seen from what follows.

The value of w lies pretty generally between the limits $1^3/_4$ and 3 pounds per square inch. Two pounds per square inch is a fair average value in English practice. Taking this value, the following table gives the value of the resistances

due to acceleration ($= \cdot 00034\ T^2 wr$) in some typical cases :—

Crank radius r in feet =		1	2	3	4
		Pounds per square inch of piston area			
Revolutions per min. T =	25	0·43	0·85	1·27	1·70
	50	1·71	3·40	5·08	6·80
	75	3·83	7·67	11·5	15·3
	100	6·80	13·6	20·3	27·2
	150	15.3	30·7	45·9	61·1
	200	27·2	54·4	81·1	—
	250	42·5	85·0	—	—
	300	61·2	—	—	—

This table shows clearly enough how at such piston speeds as were once common (200 or 250 feet per minute) the resistance due to acceleration is comparatively unimportant so long as the stroke is long. Thus for an engine of 4 feet stroke and 25 revolutions per minute (200 feet per minute of piston speed) it only amounts to 0·85 pounds per square inch. But even then it becomes appreciable if the stroke be short and the number of revolutions large, for 200 feet per minute in an engine of 1 foot stroke corresponds to 100 revolutions per minute, and therefore to 3·4 pounds per square inch of piston. At a piston speed of 400 feet per minute, which is now quite common, the resistance is 6·80 pounds per square inch for a stroke of 2 feet. For a piston speed of 800 feet per minute, which is not now regarded as unreasonable in first-class work, the resistance is as great as 13·6 pounds per square inch even for a stroke of 4 feet, while it is of course double as much, or 27·2 pounds per square inch, for a 2 feet stroke. It is obvious that for a given speed of piston (or value of the product $2\,s\,T$,—s being the

stroke) the resistance is least where the stroke s is greatest, so that from this point of view it is better to increase piston speed by increasing stroke than by increasing number of revolutions. There are not seldom cases, however, where a direct acting engine (such as, for instance, Mr. Brotherhood's "Three-cylinder" engine) has of necessity to drive very fast-running machinery without the intervention of any gearing, so that a very great number of revolutions and a very small length of stroke are necessities of the case. Here the resistance due to acceleration may be enormous. For instance, an engine having a stroke of 4 inches and running at 1,000 revolutions per minute (a piston speed of 667 feet) would have an initial accelerative resistance of 113 pounds per square inch of piston if we took the value of w the same as before, 2 pounds per square inch. In this particular case, however, the value of w is much smaller than usual, both because of the absence of piston rod and crosshead, and on account of the special care given to this matter in the design of the engine. It is probably only about 0·9 pounds per square inch of cylinder, so that the accelerative resistance is reduced to 51 pounds per square inch—still, necessarily, a very high value.

By making two or more pistons drive as many cranks placed at proper angles from one another, the disturbing effect of this accelerative resistance upon the total twisting moment on the shaft may be greatly reduced, but there are no means by which its value for each individual piston or steam cylinder can be changed.

When the engine is a vertical one there is also to be taken into account that a weight equal to that of the reciprocating parts is alternately raised and lowered during each revolution. On the down stroke this weight corresponds to an additional pressure of w pounds per square inch on the

piston, and on the up stroke to an additional resistance of the same amount. To find the true effect transmitted to the crank pin this pressure has to be added to, or the resistance deducted from, the ordinates of the indicator card along its whole length. We have assumed that the engine has a fly wheel or other rotating mass sufficiently large to keep the velocity of the crank pin sensibly uniform in spite of any irregularities in the piston pressure, so that this rising and falling weight does not make any change in the speed or acceleration of the reciprocating parts. This point will be considered in detail in the next section.

It has been assumed in the foregoing investigation that the connecting rod was infinitely long. The fact that the rod is only of finite length makes a considerable difference in the result, and may frequently have to be taken into account in practical problems. The acceleration diagram can be easily drawn from the velocity diagram (the latter found as on p. 105) in the way described on p. 200. Apart from this, however, the initial and final accelerations can very easily be found. In Fig. 153 let AB_1 be the crank and B_1P the connecting rod of an engine, in a position very near the beginning of the stroke of the piston from right to left. If the connecting rod were infinitely long we should have

$$\frac{\text{Crosshead velocity}}{\text{Crank pin velocity}} = \frac{OP_1}{OB_1}.$$

With the actual length of connecting rod the same ratio is $\dfrac{OP}{OB_1}$ instead. The velocity of the crosshead in the latter case therefore exceeds that in the former in the ratio $\dfrac{OP}{OP_1}$. By taking the angle B_1AB small enough we may assume

$AB_1 = AB$, and $PB_1 = PB$, from which (by similar triangles) we have

$$\frac{OP}{OP_1} = \frac{OA}{OB_1} = \frac{PA}{PB},$$

and if we call the length of the connecting rod $= l$, and the crank radius $= r$, we have further

$$\frac{PA}{PB} = \frac{l + r}{l} = 1 + \frac{r}{l} = \text{(say) } 1 + \frac{1}{n}.$$

The velocity of the reciprocating parts immediately after starting is therefore $\frac{1}{n}$ greater than with a connecting rod of infinite length, where n is the ratio of the length

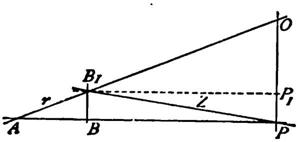


FIG. 153.

of the connecting rod to the length of the crank. The acceleration (see p. 180) varies as $\frac{v^2}{s}$, and therefore is

$$\frac{\left(1 + \frac{1}{n}\right)^2}{1 + \frac{1}{n}} = 1 + \frac{1}{n}$$ greater than before (the distance s passed through itself varying as the velocity). In the same way at the end of the stroke the final acceleration is $1 - \frac{1}{n}$ times what it would be with an infinitely long connecting rod. The first and last points of the acceleration curves can therefore be very readily drawn, and the point

where it cuts the axis is also easily found, as it lies under the point of maximum velocity (compare Fig. 150). This will now be earlier than before, that is, before mid-stroke in the direction in which the piston is travelling in Fig. 153.[1]

It would be going beyond the limits of our present work to examine in detail the influence of these accelerative resistances on the distribution of pressures in an engine. It must suffice here to give the one example of Fig. 154. Here *ABCDE* is the indicator card of an engine. Turning

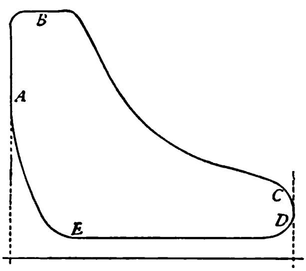


FIG. 154.

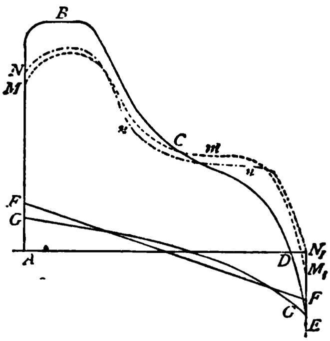


FIG. 154 A.

the lower line of the card end for end, or using the lower line of a card taken from the opposite end of the cylinder (as is necessary if we wish the ordinate of the closed curve to represent the actual pressure on the piston at any point of the stroke), and reducing the whole to a straight base, this becomes *ABCDE* in Fig. 154 A. With an infinite connecting rod the line FF_1 would be the acceleration curve, and with a

[1] The statement made by Radinger, and copied from him into many text books, that the curve of resistances due to acceleration becomes a parabola, appears to be based on quite erroneous reasoning.

connecting rod equal in length to twice the stroke GG_1 would be the acceleration curve. Subtracting the ordinates of these curves from those of the pressure curve we get the lines MM_1 and NN_1 respectively as representing the real nett effort driving the crank pin. It is necessary in these cases that the curve started with should be as nearly as possible the actual indicator diagram of the engine, and not a theoretical or conventional curve. The actual curve can be very well approximated to even before an engine is designed, and (as in the case figured) it may differ very importantly from the hyperbolic diagram.

§ 48.—FLY WHEEL.

In considering the dynamics of the ordinary steam engine in the last section it was assumed that the angular velocity of the crank shaft, and therefore also the linear velocity of the crank pin, remained always constant. This is, of course, not actually the case in practice, but means are taken to make the changes of angular velocity of the shaft so small as to be negligible, and these means we have now to consider.

Let it be supposed that for any particular engine the diagram of actual piston effort, allowing for acceleration, etc., has been found as in the last section, and that from this diagram has also been found, by the methods of § 43, the diagram of "tangential effort," or driving force acting at the crank pin and in the direction of its motion. In Fig. 155, *ABCDE* represents such a diagram drawn on a base *AE* whose length is equal to the semi-circumference of the crank pin circle, that is, to $\frac{\pi}{2}s$, if s be the stroke. We have

already seen that the *area* of this diagram must be equal to the area of the diagram of piston effort,[1] and to the area of the indicator card—for it represents precisely and absolutely the same amount of work (work done against friction only has been neglected, and this is comparatively very small in amount in such a case). We have seen also that the mean crank pin effort must correspondingly be always $\frac{2}{\pi}$ (or 0·637) times as great as the mean piston effort. The *mean* resistance of all the machines, pumps, shafting, etc., driven by the engine, reduced to the radius of the crank pin, must be exactly equal to the *mean* effort at the crank pin. It

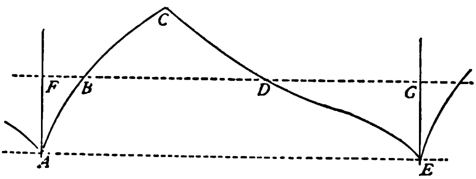


FIG. 155.

cannot be constant from minute to minute—although in some cases it is nearly so—but during any one stroke it may be assumed to be constant without much error. In Fig. 154 it is represented by $AF = EG$. The total work done *on* the crank pin, ACE, must be equal to the total work done *by* the crank pin at $AFGE$. At first the effort falls short of the resistance, until the point B is reached. Work equal to AFB must be done at the expense of the energy stored up in the rotating parts, which must in consequence be reduced in speed. From B to D the effort exceeds the resistance, the speed of rotation must increase, and work

[1] See § 43, p. 323.

represented by the area BCD is stored up in the rotating parts. After D, the point of maximum velocity (as B was that of minimum velocity), the effort again falls short of the resistance, and once more the speed falls—the area DGE representing again work done at the expense of the energy stored up in the rotating parts. The sum of the areas AFB and DGE must be equal to the area BCD if the velocity at E is equal to that at A, which is, by hypothesis, the case.

The crank pin velocity therefore fluctuates between a minimum at B and a maximum at D, and the transference of energy corresponding to this change of velocity is graphically represented by the area BCD. Calling v_1 the minimum and v_2 the maximum velocity of the crank pin, and nv_1 and nv_2 the minimum and maximum velocity of any one of the rotating masses whose weight is w pounds, and whose radius is n times that of the crank pin, then the energy stored up in this mass while the crank pin is accelerated from v_1 to v_2 is

$$\frac{w}{g}\left(\frac{n^2v_2^2 - n^2v_1^2}{2}\right)$$

$$= \frac{wn^2}{g}\left(\frac{v_2^2 - v_1^2}{2}\right)$$

and the quantity $\frac{wn^2}{g}$ is the mass of a body which, if moving at the same velocity as that of the crank pin, would take up the same work for any given acceleration as is actually taken up by the mass $\frac{w}{g}$ moving n times as fast.

The rotating parts consist of a number of different masses

having differing radii and therefore different (linear) velocities. If for each mass we find the value of

$$\frac{wn^2}{g}\left(\frac{v_2^2 - v_1^2}{2}\right)$$

taking into account the proper value of n, and then add all the quantities together, we get a sum which may be written

$$\left(\frac{v^2_2 - v_1^2}{2}\right)\Sigma\frac{wn^2}{g}$$

where $\Sigma\frac{wn^2}{g}$ is **the mass of a body which, if placed at the crank pin, would take up the same work in a given acceleration as is actually absorbed by the whole masses $\Sigma\frac{w}{g}$ each moving with its own proper velocity greater or less than that of the crank pin.** This quantity $\Sigma\frac{wn^2}{g}$ is called the **reduced inertia, or inertia reduced to the crank pin, of the rotating masses.**

The value of the whole quantity of energy given above is known from the diagram, for it is numerically equal to the area BCD, measured in foot-pounds, multiplied by the number of square inches in the cylinder area (the ordinates of BCD being pounds per square inch of cylinder area). If we are working at the problem in order to discover the fluctuations of velocity of a given engine we can compute the numerical value of the reduced inertia from the known dimensions of the engine. If this quantity be called R, and the amount of excess energy just mentioned be called E, we have

$$E = R\left(\frac{v_2^2 - v_1^2}{2}\right), \text{ or}$$

$$\frac{2E}{R} = (v_2^2 - v_1^2).$$

This does not enable us to find either v_2 or v_1 directly, but only to find either one when the other is known. The absolute speed at which the engine runs is determined by the amount of steam which the boiler can supply or is allowed to supply in a given time, and is independent of the value of E, which only fixes the *fluctuation* of speed. In this case it is the *mean* velocity which is generally known, and it is sufficiently accurate to assume that this mean velocity, v_o, is an arithmetical mean between v_1 and v_2, $\left(v_o = \frac{v_2 + v_1}{2}\right)$ in which case the whole fluctuation of velocity $v_2 - v_1 = \frac{E}{v_0 R}$. Half this fluctuation added to v_o gives the maximum velocity v_2, while the same amount subtracted from v_o gives the minimum velocity v_1.

If we are working at the problem from the opposite direction, viz., with the object of determining the reduced inertia which must be possessed by the rotating parts, in order that an engine may run within certain limits of velocity, it is treated somewhat differently. Here both v_2 and v_1 can be fixed beforehand, a somewhat more convenient as well as a more accurate mode of procedure than working from an assumed average velocity. The reduced inertia of the crank shaft, crank, balance weights, &c., is calculated, and the (comparatively very small) amount of energy which they can absorb is found. With these data we can then find what must be the additional reduced inertia of the added masses necessary to keep the velocity within the required limits. These added masses nearly always take the form of a wheel ("fly wheel") with a very heavy rim—*i.e.*, with its mass concentrated at as great a radius as possible. It is often sufficiently accurate (and errs always on the safe side) to neglect the mass of the fly wheel boss and arms, and to make

the reduced inertia of the rim alone equal to the required amount. In this case n is the ratio of the radius of inertia of the rim to the radius of the crank, and the weight of the rim, w, can be at once found.

The energy actually stored up in the rotating parts varies therefore from instant to instant, being greatest at D, where it is $\frac{v_2^2}{2}R$, and least at B, where it is only $\frac{v_1^2}{2}R$. Its *average* value, if v_o is the average velocity of the crank pin, is $\frac{v_o^2}{2}R$, which may be called E_o. The proportional fluctuation of energy per stroke is therefore $\frac{E}{E_o} = \frac{v_2^2 - v_1^2}{v_o^2}$, and if we take v_o as the arithmetical mean between v_2 and v_1, we have the fluctuation of *velocity* as

$$\frac{v_2 - v_1}{v_o} = \frac{E}{2E_o},$$

a ratio which for ordinary factory engines may be 0·04 or 0·05. It must be always remembered that this does not represent a change in the number of revolutions in one minute as compared with another, or even in one second as compared with another, but only the proportionate change (called by Rankine the *coefficient of unsteadiness*) occurring *within each revolution*, whatever time that revolution itself occupies.

If an engine has two cylinders working cranks at right angles to each other, the mean crank-pin effort must be found as in Fig. 156 by drawing the crank-pin effort diagrams of each separately, spaced half a revolution apart (as $A_1B_1C_1D_1$ . . and $A_2B_2C_2D_2$ . .) and adding the ordinates of the two curves together as in $ABCD$. . . . . The mean resistance line is AL, and the fluctuations of energy, and therefore of

velocity, occur twice as often as before, but are greatly diminished in amount. *A*, *E*, *K*, &c., are points of minimum velocity, and *C*, *G*, *L*, &c., points of maximum velocity.

It may be worth while to examine one numerical example before leaving this matter.

Let our data be an engine having one cylinder 20 inches diameter and 3 feet stroke, to work with a minimum crank

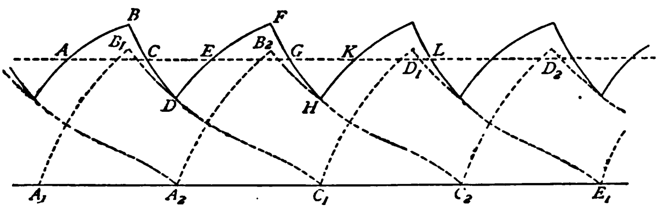


FIG. 156.

pin velocity (v_1) of 9·8 feet per second and a maximum (v_2) of 10·2 feet per second, which corresponds to a mean piston speed of about 380 feet per minute. The crank-pin effort diagram shall be that of Fig. 155, the problem being to find the weight of the fly wheel necessary to keep the velocity within the given limits. The length *BD* is 2 feet, and the mean height of the area *BCD* is 27 pounds, so that the area *BCD* represents 54 foot-pounds of energy per square inch of piston area. $\frac{v_2^2 - v_1^2}{2} = 4$, so that $54 = 4R$ and $R = 13·5$ per square inch of cylinder. The area of the cylinder being 314 square inches, the total value of the reduced inertia is $314 \times 13·5$ or 4,240.[1] The value of Σwn^2

[1] The figures here and elsewhere are rounded off for convenience sake.

must therefore be $4{,}240 \times g$, or 136,530 pounds, the weight of the mass which would be required, *at a radius equal to that of the crank pin*, to maintain a coefficient of unsteadiness of $\frac{10\cdot2 - 9\cdot8}{10}$ or 0·04. If the value of $\Sigma\, wn^2$ for the connecting rod end, crank pin, cranks, shaft, &c., be 2,000, and for the fly wheel boss and arms 4,000, there remains about 130,000 as the equivalent weight, at the crank pin, of the fly wheel rim. The radius of the crank is 1·5 feet—if we take the radius of inertia (p. 243) of the fly wheel rim as 7·5 feet, we have $n = \frac{7\cdot5}{1\cdot5} = 5$ and the actual weight of the fly wheel rim necessary will be about $\frac{130{,}000}{5^2} = 5{,}200$ pounds or 2·32 tons.

This problem may be worked backwards as an exercise, taking the weights as data along with the dimensions of the engine, the minimum velocity v_1, and the value of the area BCD, and finding the maximum velocity v_2.

§ 49.—CONNECTING ROD.

In the fly wheel we have examined the kinetic relations of a body rotating about a fixed centre, in the pumping engine the case of bodies having only motion of translation, and we may now, finally, examine the case of the acceleration of a body having general plane motion. The connecting rod of a steam engine is a very familiar case of such a body. We have fully examined its motion in § 12 and elsewhere, and need say nothing about it here; it turns about continually changing virtual centres, and we have the means of

finding with the utmost ease the particular point about which it is turning at any instant. It undergoes alternate positive and negative acceleration, just as does the crank shaft and fly wheel, and therefore alternately absorbs and restores energy. Its motion does not, therefore, reduce the amount of available work done by the steam, but only alters the distribution of effort on the crank pin, just as the acceleration of the piston and crosshead alter the distribution of piston effort (§ 47).

It is commonly assumed that the alterations due to the connecting rod may be taken to be the same as if half the rod were attached to the reciprocating parts, and formed part of them, and the other half revolved steadily with the

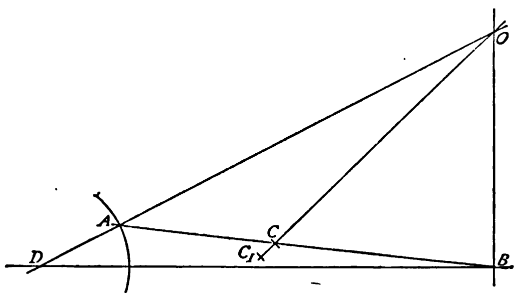


FIG. 157.

crank pin, forming virtually a part of the fly wheel or revolving masses. The error caused by these assumptions is not in general important; we shall here work out our results in the first instance without any such assumption, and then compare them with the approximation obtained by the ordinary method.

Let us suppose AB (Fig. 157) to be a connecting rod for

whose motion at present the point O is the virtual centre,[1] and whose mass-centre is at C. Adding together the moment of inertia of the rod about C and that of the rod supposed concentrated at C about O (see § 32) we can find the radius of inertia OC_1, at any point at which radius we may suppose the whole body to be concentrated. The velocity of any such point (as C_1) is of course proportional to its radius. The velocity at the point A is assumed constant and known, the velocity at C_1 is therefore greater (or less) in the ratio $\frac{OC_1}{OA}$, so that it can be found at once.

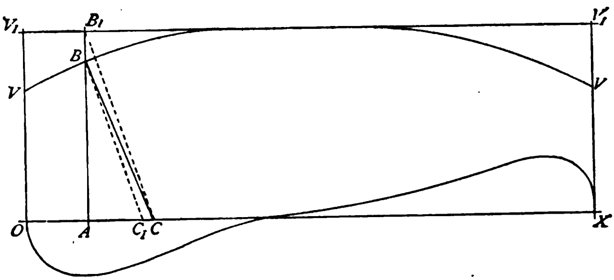


FIG. 158.

In Fig. 158 let OX be the distance traversed by the crank pin during one revolution, and OV_1 its constant velocity (= 10 feet per second), so that the straight line $V_1\ V_1$ is a velocity diagram for the crank pin on a distance base. Find values of OC_1 (Fig. 157) for different positions of the connecting rod, and from the assumed constant velocity of A find the

[1] In the example given here the following are the proportions taken: $DA = 1{\cdot}5$ feet, $AB = 6$ feet, $AC = 2{\cdot}4$ feet, the weight of the rod 700 pounds, the (constant) velocity of the crank pin 10 feet per second, the moment of inertia of the rod about an axis through its own mass centre 63.

velocity (p. 87, Fig. 37) of points at the radius C_1, any one of which may represent the connecting rod for the time being. Set off a curve of these velocities as $V\,V$ in Fig. 158. This curve does *not* give successive velocities of any one point in the way in which the line $V_1\ V_1$ represents the velocities of the point A. It gives the successive velocities of the successive points which in turn come to be at a distance equal to the radius of inertia from the virtual centre. Let AB (in Fig. 158) be this representative velocity of the rod at any instant, then AC will be its acceleration at that instant (p. 200). AC will therefore represent on some scale the resistance due to acceleration. Up to the highest point of the curve the resistance acts *against* the engine's motion, after that (while energy is being *restored*) it acts *with* the engine, the alternation being exactly similar to the one we have examined in § 48. AC represents also on some scale, as we know, the force necessary to produce the given acceleration if applied at the point accelerated. But actually the force is applied at a radius greater in the ratio $\frac{A\,B_1}{A\,B}$ (for the radii are proportional to the velocities) and must therefore be smaller in the same ratio. This can be found at once by drawing BC_1 parallel to B_1C_1 (which need not be drawn), which gives AC_1 as the force acting *at the crank pin* A, either from the pin on the connecting rod or from the connecting rod on the pin, according to whether the velocity of the rod is increasing or decreasing. The corresponding force at the crosshead or piston requires to be found from this by any of the methods already given.

The scale on which $A\ C_1$ is to be measured is to be found exactly as in § 28. If the scales of velocity and distance are equal, the scale of acceleration will be the same. If there are n times as many units of velocity per inch on the

paper as there are units of distance, there must then be n times as many units of acceleration per inch as there are units of velocity. The diagram Fig. 157 is, for instance, drawn with a velocity scale of 10 feet per second to the inch and a distance scale of 1 foot per inch; n is therefore $= 10$, and the acceleration scale is 100 foot-seconds per second to the inch.

For purposes of comparison with such figures as those of § 48 we require not only to read this acceleration as force, but as force per unit of piston area: $f = \frac{m}{A} a = \frac{w a}{g A}$, if m be the mass of the connecting-rod, a the acceleration, and A the area of the piston. The distance $A C_1$, therefore, if it is to be read as equivalent pressure at the crank pin per unit of piston area, must be read on a scale having $\frac{m}{A}$ or $\frac{w}{g A}$ times as many pounds per inch as the acceleration scale has units per inch. In the present case the weight of the connecting rod has been taken as 700 pounds, its mass is therefore 22. The piston for such a rod might be 40 inches diameter, for which $A = 1{,}256$. The force scale would therefore be $100 \times \frac{22}{1{,}256} = 1{\cdot}75$ pounds per square inch of piston area per inch. The curve drawn about the axis OX in Fig. 158 shows the resistance *at the crank pin* on this scale, its ordinates are simply values of $A C_1$ set down or up. Worked back to the piston these give the quantities shown in Fig. 159, where it will be seen that the maximum value of the resistance due to the acceleration of the connecting rod in this case is under 1 pound per square inch of piston area. If the ordinary assumption were made that half the rod was attached to the crosshead, and moved with it, the effect on

the piston, neglecting the obliquity of the connecting rod, would be that shown by the straight line in the last figure, whose end ordinate is .00034 $T^2 wr$ pounds per square inch (§ 47), which is here equal to 1·17.

We have seen in § 26 that all particles of a rotating body do not possess the same radial acceleration, because although they all have the same angular velocity they have very different radii. But a difference of radial acceleration between two particles must be accompanied by a difference of centripetal or centrifugal force. This difference is itself a force, tending to alter the form of the material, and is balanced by stresses

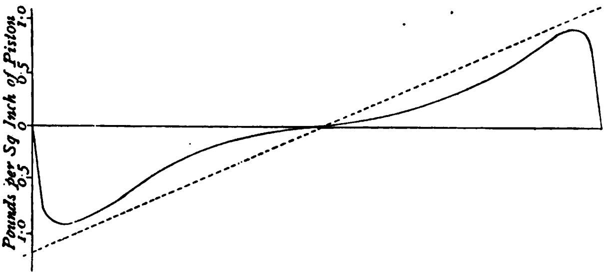


FIG. 159.

within the material itself, resisting that alteration of form. In the case of a fly wheel this difference of centrifugal force has occasionally become so great as to require, to balance it, stresses greater than the material was capable of exerting. In this case the wheel has broken to pieces—one of the most serious accidents which can happen in a factory. A similar accident, although a less serious one, occurs sometimes with a connecting rod. The effect is generally that the rod is bent in its plane of motion, and sometimes broken, and of course damage, more or less serious, is done to the

engine of which it forms a part. In designing the connecting rod for high-speed engines it is often necessary to take into account the bending stress which may be thus caused in the rod, and which may form a notable addition to the direct stress caused by the alternate push and pull of the crosshead. The determination of such stresses does not, however, fall within the scope of this work.

§ 50.—GOVERNORS.

The ordinary "rotating pendulum" governor forms an excellent example of a number of points treated in Chapter VII., and for this reason, as well as for its intrinsic importance as part of an engine, we shall here examine it in some detail. Let us take first the simple Watt governor, such as is sketched in Fig. 160 and investigate the conditions under which it works. When the governor is revolving at a uniform speed the ball remains always at the same distance from the spindle. As regards motion up or down it must therefore be in static equilibrium. This equilibrium is brought about by the balancing of three forces,[1] viz. the weight of the ball, which acts always downwards, the "centrifugal force" (p. 227) which acts always horizontally outwards, and the stress—tension—in the ball arm, or rod by which the ball is suspended. The direction of this third force depends on the slope of the arm, which varies for every position of the ball, but under all circumstances, when the ball is in equili-

[1] We shall here neglect the weight of the ball arm, as well as all frictional resistances of the sleeve, &c. (as to which see Fig. 170), and shall suppose that the mass of the ball may be concentrated at its mass-centre.

brium, it must be such that the stress is exactly equal to the sum of the other two forces.[1] The downward force depends on the weight alone, and may be written w. The centrifugal force depends on the mass, its velocity, and its radius, and is equal (p. 228) to $\frac{wv^2}{gr}$, for which we may write c. The stress in the ball arm must therefore be represented in magnitude and direction by the third side of a triangle of which the two sides c and w are given. Such a triangle is shown in $A\,C\,D$ in Fig. 160. The slope of $O\,A$, the ball arm,

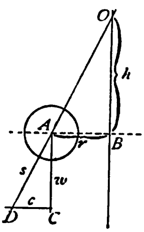


FIG. 160.

must be the same as that of the line $A\,D$; the position of the ball is therefore absolutely determined, for the radius $A\,B$ has been already assumed in calculating the value of c. The distance $O\,B$ – the vertical distance between the ball centre and the point where its line of suspension cuts the vertical axis—is called the *height* of the governor, and may

[1] This stress is, of course, the sum of the other two forces passing through the virtual centre about which the ball arm can swing. If the arm is bent, its direction is not that of the axis of the arm, but of a line joining the centre of the ball and the centre of the pin about which it swings.

be written h. If we further write r for the radius AB, we have, at once, by similarity of triangles

$$\frac{h}{r} = \frac{w}{c} = \frac{w}{\dfrac{w\,v^2}{gr}} = \frac{gr}{v^2}$$

(r being measured in *feet* and v in feet-per-second, as usual) from which

$$h = g\,\frac{r^2}{v^2}.$$

If now we write T for the number of turns *per second* made by the governor,

$$v = 2\,\pi\,r\,T$$

and by substitution

$$h = \frac{0{\cdot}8154}{T^2}\text{ feet or }\frac{9{\cdot}785}{T^2}\text{ inches.}$$ [1]

It thus appears that **the height of a governor of this kind depends absolutely upon its angular velocity.** No matter what the size of the engine, or the mass of the balls, or their radius, such a governor rotating 60 times per minute ($T = 1$) must have a height neither more nor less than 9·78 inches, and for any other speed the height will vary inversely as the square of the velocity. It is obvious that this places a practical limit to the use of such a governor as this, because at even very moderate speeds its height becomes so small as to be constructively very inconvenient. Thus a Watt governor running at even 100 revolutions per minute could only have a height of $\frac{9{\cdot}785}{1{\cdot}66^2} = 3{\cdot}58$ inches—far too small a height to be practicable except for extremely small engines.

[1] As some absolute value of g is included in the calculation here, this height is correct only for places when this particular value of g is the right one. No practical error, of course, is introduced by the small variations of g at any place where an engine would be placed.

It will be seen that the height h varies with every position of the ball, being greatest when the engine is at rest. Let us suppose an engine starting from rest, and see what happens to the governor. The weight w (see Fig. 161) is not altered by the fact that the engine is at rest, and there must be *some* stress AD in the ball arm. There must therefore be some horizontal force CD. There can, however, be no centrifugal force when the engine is at rest, and CD is simply the pressure of the spindle or some other part of the governor against the ball. The engine starts and rotates with gradually increasing speed. As the radial acceleration increases

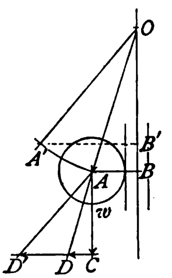


FIG. 161.

so does the centrifugal force, which is at first very small—less than CD. The ball arm therefore does not rise, but its pressure against the spindle diminishes by an amount exactly equal to the centrifugal force. When the centrifugal force has become equal to CD, the pressure on the spindle has become zero. The ball arm is held in static equilibrium in its sloping position not by pressure against the spindle, but by the (at this moment) exactly equal "centrifugal force" corresponding to the radial acceleration which it has received. If the speed now continue to increase, CD becomes greater,

say CD'. But AC, the weight, does not increase. Therefore the ball arm is no longer in equilibrium in its present position, for AD', the sum of AC and CD', is not in the direction OA. The ball will move outwards until the arm reaches a new position OA' parallel to AD', and will then (if the speed, and with it the centrifugal force, remain the same) rotate steadily in that position, with radius $A'B'$ and height OB', the latter bearing the particular relation to the speed which we have already examined.

Of course the ball could be forcibly compelled, by a sufficiently strong fastening, to remain in the position A at the speed corresponding to A'. The radial pull in such a fastening would be $D'D$. The mechanism would then, however, cease to have any action as a governor, so that the case is not one which we require to consider.

It appears then, apart from this fixing up of the governor, (which would of course render it useless for our purposes) that at any given speed there is only one height at which the balls of a rotating governor can remain steady as they rotate, and that this height is absolutely determined by the angular velocity or speed of rotation. The object of the governor is to keep the speed of the engine as constant as possible, and this is brought about by connecting the balls (through certain intermediate mechanism) to a throttle or a cut-off valve in such a way that when the resistance on the engine (§ 47) diminishes (and its speed therefore increases) the effort on the piston due to the steam pressure may be caused to diminish also, and so prevent the increase of speed being too great. It is to be specially noted that this governor cannot *prevent* increase of speed—at best it can only *limit* it. For its action in lessening the driving effort—whether by throttling the steam or in any other way, is entirely brought about by some change in the position of

the balls, and any such change is inevitably accompanied by *some* alteration of height, and therefore of speed.

The practical point is therefore to arrange that this inevitable variation in speed should be kept within certain limits fixed beforehand. Let $\frac{1}{a}$ be the fraction of the mean speed equal to the utmost allowable variation between the intended maximum and minimum speeds, or what may be called the *fluctuation* in speed. Let h be the height of the governor when running at mean speed, and Δh the extreme allowable variation of height corresponding to the allowable variation in speed. If v be the mean velocity, the maximum velocity will be $v\left(1 + \frac{1}{2a}\right)$ and the minimum velocity $v\left(1 - \frac{1}{2a}\right)$. If we write v_2 and v_1 for these quantities respectively, and h_2 and h_1 for the corresponding heights of the governor, then

$$h_1 = h\,\frac{v^2}{{v_1}^2} = \frac{h}{\left(1 - \frac{1}{2a}\right)^2}$$

$$h_2 = h\,\frac{v^2}{{v_2}^2} = \frac{h}{\left(1 + \frac{1}{2a}\right)^2}$$

$$\text{and } \Delta h = h_1 - h_2 = \frac{h}{\left(1 - \frac{1}{2a}\right)^2} - \frac{h}{\left(1 + \frac{1}{2a}\right)^2}$$

from which we obtain the ratio of change of height to height corresponding to mean velocity,

$$= \frac{\Delta h}{h} = \frac{2}{a\left(1 - \frac{1}{2a^2} + \frac{1}{16a^4}\right)} = \frac{2}{a} \text{ very nearly.}$$

If we had taken $\dfrac{1}{1-\dfrac{1}{2a}}$ as equal to $\left(1+\dfrac{1}{\dfrac{1}{2a}}\right)$ as is usually done here, we should have had $\dfrac{\Delta h}{h} = \dfrac{2}{a}$ exactly, and the error in doing this is very small.

The ratio of change of height to mean height, or what we may call the fluctuation in height, is therefore equal to twice the fluctuation in speed. If, for instance, we require an engine to work within a fluctuation of 5 per cent. in speed, or $\dfrac{1}{20}$, the ratio $\dfrac{\Delta h}{h}$ would be $\dfrac{2}{20}$ or $\dfrac{1}{10}$. If h therefore were 10 inches we should have to arrange the governor mechanism so that a vertical rise of not more than one inch in the balls should be sufficient to control the working of the throttle valve through all its range of action. It is thus not to be wondered at that the controlling action of such governors is often very defective indeed.

What is practically wanted is some means for increasing the height "due to" a particular speed, so that the given *absolute* variation in height required to effect the proper motion of the throttle valve or expansion valve should correspond to a smaller *proportionate* variation in speed. This is easily done. Let Fig. 162 represent the position of a governor arm running steadily with weight AC and centrifugal force CD as before. Now let a weight be placed on the ball, but so arranged that its mass centre lies upon the vertical axis OB. A mass so arranged has no centrifugal force, and therefore adds nothing to CD—but the action of its weight is not affected by its position, so that it adds CC' to the vertical force acting on the ball. As a consequence the magnitude and direction of the balancing stress in the

ball arm becomes $A D'$, and the arm must assume the position $O A'$, parallel to $A D'$. The additional weight has therefore caused the governor to run at the increased height $O B'$ instead of $O B$, without changing its velocity. The ratio $\frac{O B'}{O B} = \frac{A C'}{A C}$; that is, the increase of the height is in direct proportion to the increase of weight. The alteration in height required to work the throttle or other valve bears of

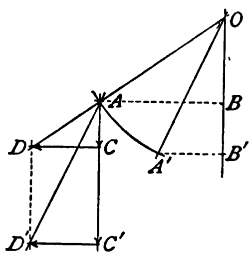


FIG. 162.

course a correspondingly reduced ratio to the total height, that is, $\frac{\Delta h}{h}$ becomes smaller,—Δh being constant and h increased,—the fluctuation in velocity $\frac{1}{a}$ has therefore become smaller in the same ratio. This is called increasing the **sensitiveness** of the governor, **the measure of sensitiveness being the smallness of the alteration in velocity which corresponds to a given absolute alteration in height.** A governor of this kind is called a **loaded** governor.

It has been supposed that the weight has been added as in Fig. 163, so that its rise or fall has been exactly equal to

that of the balls. In this case CC' in the last figure must have been taken either the whole or the half of the added weight, according as AC was the weight of one ball or of the pair of balls. In practice, as in the "Porter" Governor, the weight is more commonly added as sketched in Fig. 164. Here the linkwork connection of the balls, the weight, and the spindle is such that the weight always moves through *twice* the vertical distance moved through by the balls. Any weight W, therefore, attached in this way is equivalent to $2W$ resting on the balls direct as in Fig. 163, and the distance CC' in Fig. 162 must be equal to the whole added

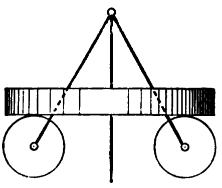


FIG. 163.

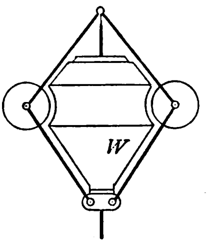


FIG. 164.

weight if AC is the weight of one ball, or *twice* the added weight if AC is the weight of two balls.

We have seen the change of height with load on the governor; before leaving this matter it may be well to look at the change of velocity due to load if the height is kept constant. In Fig. 165 OA and OA' are positions of the ball arms of two governors, one unloaded, the other with a load CC'. The centrifugal force $\frac{wv^2}{gr}$ is the same in both cases, $= CD = C'D'$, but its component factors have altered. Writing $\frac{w}{g}\ 4\pi^2T^2r$ (as on page 372) for the centrifugal

force, we see that $\frac{w}{g}$ $4\ \pi^2$ must be unchanged. The remainder T^2r, therefore, must also be constant, from which it is evident that T^2 must vary inversely as r. This latter quantity is reduced in the ratio $\frac{A'B}{AB}$, which is easily seen to be equal to $\frac{AC}{AC'}$. We therefore have at once the conclusion that the square of the angular velocity has increased in the direct ratio of the weights, so that the speed corresponding to any height has increased as $\sqrt{AC'} : \sqrt{AC}$.

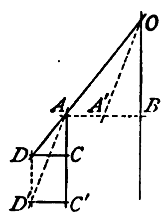


FIG. 165.

By loading a governor its sensitiveness may be increased to almost any required extent, but it can never become **isochronous**, for *some* variation of height is unavoidable, and a governor to be truly isochronous must have the same height for every position of the balls. Such a governor would run steadily at only one speed, for however much the position of the ball arms changed, or whatever action had taken place on the throttle valve, the height would remain the same, and the engine, as soon as the temporary acceleration had ended, would be running at the same speed as before, and not, as in all the cases we have been considering,

at a somewhat greater or less speed. If for instance the ball centre could be compelled to move in an arc of a parabola, and the ball arm to be always normal to that arc, the height of the governor would be the sub-normal (see p. 203) to the parabola, and this distance is constant. Truly parabolic governors have been constructed, but in general the plan used is that sketched in Fig. 166, which represents what is known as a **crossed-arm governor.** Here the ball centres move in a circular arc just as before, but the centre of the arc is so chosen that the arc itself nearly coincides

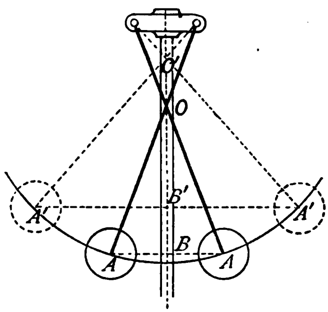


FIG. 166.

with a parabolic arc. To design such a governor the height should first be determined for the particular speed required, —a parabola should then be drawn having this height for its sub-normal, and having the axis of the governor spindle for its axis. Choosing then the part of the parabola most convenient for the swing of the balls, normals should be drawn from its end points. The intersection of these normals gives a centre for a circular arc approximating closely to the desired curve.

While an arrangement like this makes the height of a

governor constant (or nearly so) it does not, of course, alter the relation between speed and height, which remains as before. Constructively, however, a smaller height can be used with much less inconvenience than in the former case; because the length of the ball arm is proportionately greater. But it may still often happen that the height is inconveniently small for practical purposes, and in this case the same expedient may be resorted to as before—the addition of further mass, placed so that its centre lies in the axis of revolution. Here of course there is no question of increasing sensitiveness, because there is no change of height, but the height corresponding to any speed is made greater, or the speed corresponding to any height less, in the same way as before.

If a governor were perfectly isochronous, the least change of speed could cause it to move up or down through its whole range. As such changes are constantly occurring, the governor would be continually shifting, or *hunting*, as it is called. As this would be troublesome, the governor is either made to have some small change in height, or else some artificial resistance is arranged to come into play when the governor moves, in such a way as to make it more sluggish or less sensitive to minute changes of speed in the engine. Into this matter we cannot, however, enter here, further than to note that any arrangement of this kind is said to increase the **stability** of the governor.

There are a number of governors made which are approximately isochronous, but which are not really pendulum governors, and in which the isochronism is not attained by keeping the height constant, but by so constructing the governor that the relation of the speed to the height alters as the height itself changes, and that the one change is made approximately equal and opposite to the other, the speed in

this way remaining the same notwithstanding the change of height. As the working of these governors is perhaps less easy to understand than that of ordinary pendulum governors, and is equally important and interesting dynamically, we shall examine two of them.

Fig. 167 is a sketch of a governor designed by Mr. Wilson Hartnell of Leeds, which not only can be given any required degree of isochronousness, but has also the advantage, to be presently discussed, of great **powerfulness**. The place of

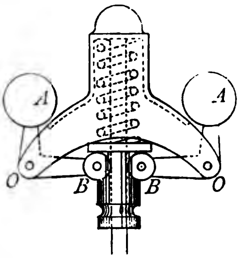


FIG. 167.

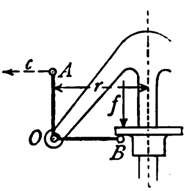


FIG. 168.

the ball arm is taken by a right-angled equilateral bell-crank lever, on one arm of which, OA, is a weight, while the other, OB, presses against a spiral spring. The conditions as to equilibrium are sketched in Fig. 168. The governor will run steadily in the position shown if the moment of c, the centrifugal force, about O is equal to that of f, the pressure of the spring; that is, if

$$c.\,OA = f.OB.$$

As the two arms OA and OB are made equal this is simply saying that c must equal f. But $c = \frac{wv^2}{gr}$, in which $\frac{w}{g}$ is constant, and $\frac{v^2}{r}$ is proportional to $\frac{r^2}{r}$ or simply to r. The

centrifugal force therefore increases directly as the radius, and to make the governor run steadily in any position at the same speed it is only necessary so to adjust the pressure of the spiral spring that it may increase proportionately (as it is compressed by B) to the increase of radius. For while the centrifugal force has varied as the radius, the velocity v (linear velocity of the ball) has also varied as the radius, and therefore the number of revolutions, or angular velocity, has remained unchanged. Or this may be put in another way; writing $c = \frac{4\,w\,\pi^2\,r\,T^2}{g}$ as before, we have $\frac{4w\pi^2}{g}$ constant and c varying as rT^2. But we make the variation of r equal to that of c, so that T^2,—and therefore of course T, the number of revolutions per second—remains unchanged.

When the weight arm, OA, is not vertical, the weight itself has some moment about O. This moment is always small because of its small leverage, but there is no difficulty in taking it into account, if necessary, in adjusting the spring pressures.

Fig. 169 shows an outline of Dr. Pröll's governor, a type much used in Germany, belonging to the class of approximately isochronous governors which we are now considering. Here the ball arm, AB, does not turn about a fixed point on the governor spindle, but forms in reality one link of a slider-crank chain, having its virtual centre at the point O. The weight w is attached to AB, but not in the line of its main axis. The motion of AB is constrained by the link CB, which turns about C, and the slider or sleeve, DA, which slides up and down on the governor spindle. On this slider rests a weight w_1, which however is not essential to the governor, but used merely for the purposes formerly described.

In this case the ball arm, when running steadily in any position, is in equilibrium under five forces, viz. the weights w and $\frac{w_1}{2}$ (the other half of w_1 being taken by the other arm of the governor) the stresses in the links CB and DA, and the centrifugal force. The two stresses along CB and DA both pass through the virtual centre, O, of the ball arm, and therefore, do not tend to move it (§ 36). The other three forces must then be together equal to zero, or their

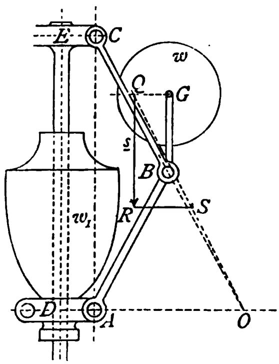


FIG. 169.

sum must pass through the virtual centre (§ 39). To find the speed at which the governor will run steadily, which we may suppose to be the object of our calculation, we first add together $\frac{w_1}{2}$ and w, and find the position of their sum s. (The construction for this is omitted; it must be remembered in making it that $\frac{w_1}{2}$ acts on the ball arm at A, not at D.) The centrifugal force is horizontal and may be taken as acting through the centre of w. The sum of the weights and

the centrifugal force must, therefore, pass through the point Q. But it must also pass through O. We need, therefore, only draw $Q\,O$, and set off $\frac{w_1}{2} + w$ vertically as $Q\,R$, and we can at once draw the horizontal $R\,S$, which gives us the centrifugal force on the same scale as that used for the weights. We have then

$$RS = \frac{4\pi^2 w T^2 r}{g},$$ [1]

of which all the quantities are known except T, which can, therefore, be found at once.

It is not possible here to go into the method of proportioning a governor such as this; we can only say that there is no difficulty in arranging it (as has been done in the proportions shown in the figure), so that the centrifugal force increases as the radius exactly as in the last case, the speed of the governor, therefore, remaining constant.[2]

The governor furnishes us with still another dynamic problem of importance and interest, which we have now the means of solving. It often happens that the action of the governor upon the speed of the engine is very indirect, and takes place through the intervention of mechanism offering a very great resistance to any change of position of the governor sleeve. This resistance of course comes into play

[1] It must be remembered here and elsewhere that r is always to be taken in *feet*, not in inches.

[2] As an exercise the centrifugal force and speed of the Pröll's governor may be worked out in different positions with the following proportions:—DE (lowest position) 14 ins.; $EC = DA = 2$ ins.; $CB = AB = 8$ ins.; $BG = 4$ ins.; angle $ABG = 120°$; $w_1 = 40$ pounds; w (each weight) $= 25$ pounds. It will be found that the governor will run steadily at a speed varying little from seventy-four revolutions per minute.

only when the governor tends to alter its position, not when it is running steadily. The relations of speed to height, and so forth, remain exactly as we have already found them. An alteration of speed is not, however, at once accompanied by an alteration of height, as we have assumed it to be. Thus let Fig. 170 represent a governor of this kind, running steadily with such a speed that CD represents the centrifugal force, and let AC represent the weight of the balls, and CC' the resistance that has to be overcome[1] at the ball centres before the sleeve can lift. Then the centrifugal force must

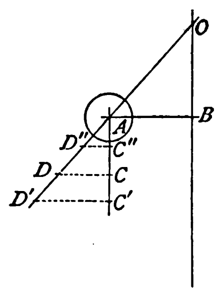


FIG. 170.

increase from CD to $C'D'$ before the ball arm can begin to move upwards. As here the radius remains unchanged, the only factor in the centrifugal force which can alter is the angular velocity. If we call T the original number of turns per second, and T_1 the number to which the engine's speed must rise before the governor balls can move under the given

[1] This resistance may be greater or less than the actual resistance to the motion of the sleeve. In the case of a Porter governor (Fig. 164) it would be double as much, for reasons which we have already seen.

resistance, and before, therefore, the governor can do anything to check the rise of speed, we shall have

$$\frac{T_1^2}{T^2} = \frac{D'C'}{DC} = \frac{AC'}{AC}$$

$$\text{or } T_1 = T\sqrt{\frac{AC'}{AC}} \text{ turns per second.}$$

Of course a change of precisely similar nature will occur when the engine speed falls, only the resistance CC' will then act (as CC'') upwards, and against the weight of the balls instead of with it. The speed of the governor must therefore fall to

$$T_2 = T\sqrt{\frac{AC''}{AC}} \text{ turns per second.}$$

before the governor commences to act.

A governor is said to be **powerful** in proportion to the smallness of the change of speed produced by a given resistance. Clearly the power of a governor depends upon the ratio of the normal downward pressure AC, to the added resistance CC' or CC''. Besides the advantages already examined, therefore, a loaded governor has a very great advantage in point of "powerfulness," or ability to move promptly against resistance. Some resistance must, however, exist in all cases, and within the limits caused by it the governor cannot even begin to have any action. That is to say, unless a governor and all its mechanism could work without even so much as frictional resistance to the motion of its sleeve, there must be some range of speed corresponding to the difference between the forces $C'D'$ and $C''D''$, within which the engine may vary without moving the governor, and therefore without receiving any control from it. For example let the resistance $CC' = CC'' =$ ten pounds,

and the force $A\,C$ = forty pounds, then the ratio in which the speed may increase before the governor can act will be

$$\sqrt{\frac{50}{40}} = 1 \cdot 12 \text{ or } 12\% \text{ increase,}$$

and the ratio in which the speed may similarly diminish will be

$$\sqrt{\frac{40}{30}} = 1 \cdot 15 \text{ or } 15\% \text{ decrease.}$$

If besides the ball weights there were an additional weight centred on the spindle of say 80 pounds, these quantities could be reduced to 4 per cent. and 4·4 per cent. respectively. If the resistance remained steady at its assumed value, the speed might be *permanently* increased or decreased within these limits, the governor remaining unaffected, and therefore useless. In practice, however, the resistance always varies greatly during each stroke, according to the relative positions of the different parts of the mechanism. If at any instant—in the example supposed—it should be *greater* than ten pounds nothing different would happen, but if at any moment it should fall short of ten pounds, a portion of the centrifugal force $C'D'$ would be unbalanced, the ball arms would cease to be in equilibrium, the balls would move upwards and the governor regain control, or partial control, over the speed. In examining the effect of this kind of resistance on the speed, its minimum value is, therefore, of more importance than its mean. This intermittent controlling action can often be seen by carefully watching the working of a governor which regulates the cut-off by moving a block in a vibrating slotted link, such for example as is shown in Mr. Arthur Rigg's *Steam Engine*, plate 47.

In respect to the promptness of action of a governor, it has also to be remembered that, quite apart from extraneous

resistances such as those just discussed, the added centrifugal force DD' (Fig. 161) or $C'D' - CD$ (Fig. 170) has to set in motion the masses of the balls and counterweight. The speed with which the governor can act will therefore (other things being equal) be determined by the acceleration which the added centrifugal force can produce in these masses. (It must not be forgotten that the increase of centrifugal force does not itself occur instantaneously.) The same cause may produce a more important result in the overrunning by the balls, in consequence of the acceleration which they have received, of their proper final positions of (static) equilibrium, an effect analogous to the "hunting" action already mentioned.

In concluding this section, it may be worth while to distinguish between the actions of a governor and a fly-wheel in controlling the speed of an engine. An engine works with a continually varying effort against a continually varying resistance, the variations of effort and of resistance being quite independent of one another. The fly-wheel acts as an equaliser between these, storing up and restoring energy in turn so as to keep the engine running as uniformly as possible so long as the mean effort is equal to the mean resistance. But the fly-wheel can do nothing to make the mean effort equal to the mean resistance, nor will it run more steadily at any one speed than at any other. If the mean resistance to the motion of the engine increases, for example, and remains permanently greater than the mean effort, the fly-wheel will keep the engine going by making up the difference from its own store of energy until it ceases to have any, that is until the engine stops, unless indeed the resistance diminishes as the speed decreases, so as to allow the engine to run steadily again at some lower speed. A governor, on the other hand, acts at best only once in a stroke, by altering the amount or

the pressure of steam entering the cylinder. It therefore has no effect whatever in steadying the engine under the continual fluctuations of velocity due to the ever recurring differences between effort and resistance. If, on the other hand, the mean resistance becomes permanently greater or less, it can permanently increase or diminish the mean effort so as again to cause equality between mean effort and mean resistance. The governor, therefore, determines the mean speed at which an engine shall run,—the engine must either run at that speed or remain quite uncontrolled by the governor,—and it makes the mean effort equal to the mean resistance at that speed. The fly-wheel determines the degree of steadiness with which the engine shall run at the speed fixed by the governor, so far as that steadiness is affected by instantaneous differences between an effort and a resistance whose average values are otherwise compelled to be equal.

CHAPTER X.

MISCELLANEOUS MECHANISMS.

§ 51.—THE "SIMPLE MACHINES."

In the older books on Mechanics, before the development of the system of machine analysis which we have used, and which is essentially due to Professor Reuleaux, the actual machines of the engineer are generally taken as being represented by certain combinations called "simple machines." Out of these, as elements, it is more or less consistently assumed that actual machines are built up. It is not worth while here to discuss a theory so hopelessly inconsistent with facts as this.[1] It will be right, however, to notice what is the real position of the "simple machines" as mechanisms.

The **lever** and the **wheel** and **axle** are, in reality, kinematically identical. Each is often figured in an impossible fashion, the lever as a bar, resting quite unconstrainedly on a triangular fulcrum, the wheel and axle as a single body poised, unsupported, in mid air. To form part

[1] It is a matter for great regret that the study of these "simple machines" should still be sanctioned and encouraged by the examinations of the University of London, which have so important an influence on the direction of teaching in their own subjects.

of a mechanism or machine, the desired motion of each must, as we know, be constrained, and this motion is nothing more than a rotation about a fixed axis. In a complete form, therefore, lever and wheel and axle are neither more nor less than **turning pairs,** such as we first examined in § 10.

The **inclined plane** is also usually drawn in an unconstrained form, being pictured as an irregular lump of material resting on a wedge-shaped block. Here the essential part of the motion is the sliding of the one body upon another, the slope or incline is entirely accidental. The motion imperfectly represented by the inclined plane is simply that obtained in complete constrainment by the **sliding pair.**

FIG. 171.

The **wedge** is a much more complex combination than either of the three "simple machines" just examined. It is not uncommonly pictured as in Fig. 171, which shows, it is needless to say, an unconstrained combination not representing any possible part of a machine. The constrained motions which are usually assumed to belong to a wedge are those of the mechanism shown in Fig. 172, which contains three links, each connected with the other two by a sliding pair, the pairs being marked 1, 2, and 3 in the figure. This chain deserves notice in several respects, for it differs

somewhat from any we have hitherto examined. In the first place, the chain is not in itself constrained, either link can be moved in one direction without affecting the others. In the cases where it is used in practice, its constrainment is effected by means of external forces, acting as shown by the arrows, Fig. 172, and caused to act permanently so long as the mechanism is used. For this reason the mechanism is said to be *force-closed*, **force-closure** meaning the constrainment, or closure, of a chain or a pair of elements by

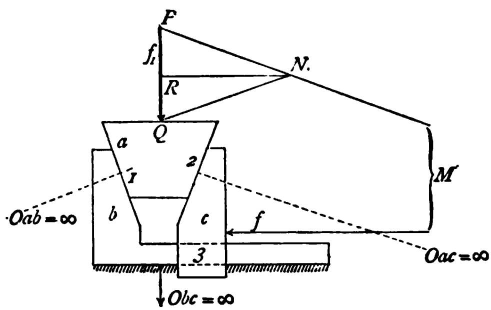


FIG. 172.

external force in the case where the proper kinematic pairing is incomplete. Another characteristic of the chain is that all its three virtual centres are at infinity. They must still be three points on one line, but this line is now the "line at infinity." All the constructions given in former sections can be equally well carried out here. For example, let f_1 be a force acting on a, to find the balancing force in the given direction f upon c, the link b being taken as the fixed link. We first resolve f_1 through the points O_{ab} and O_{ac}, that is, through the fixed point of a, and the common point of a and c. This is, of course, just as easy

as if those points were not inaccessible, for we know their directions, and can at once find the two components PN and NQ. We neglect NQ because it acts, by hypothesis, through the fixed point of a, and therefore does not require balancing. We have then only to resolve PN in the direction of f and at right angles to it, to obtain NR, the force required. For this is the same as our former construction, viz., to resolve PN through a point M, where its direction joins that of f, and through the fixed point of c, viz., O_{bc}.

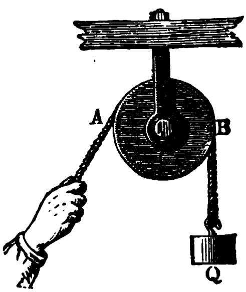


FIG. 173.

The **pulley** in various forms, such as that of Fig. 173, often appears in the list of simple machines. In the form sketched it is a chain of three links (cord, pulley and frame) and is not only a force-closed chain, but one in which a non-rigid, or simply *resistant* (see § 61) link is used. It is thus a case of very considerable complexity, and one only available for a machine under certain special conditions. It will be further considered in § 61 along with other mechanisms of the same type.

The **screw**, lastly, forms—with its nut—merely a pair of twisting elements (p. 58), constraining a motion which

is no longer plane, and which will be more fully examined in § 62.

The so-called **funicular machine**, a cord fixed at both ends and loaded with isolated weights, which is sometimes included among the simple machines or "mechanical powers," requires no notice here, as it is not a machine at all, but merely a skeleton form of **structure.** Its various segments are not intended to have any motions whatever relatively to each other, constrained or otherwise. It is merely the concrete representation of the *link polygon*, and the type form of certain most important structures, such as the suspension bridge and the bowstring girder.

§ 52.—ALTERED MECHANISMS. EXPANSION OF ELEMENTS.

WE have seen that the form or shape given to the body of the link of a mechanism is of no importance in connection with the movements of the mechanism, so long as it does not impede those movements in any way. We have now to notice further that although the *form of the elements* connecting the links is vital to the mechanism, their *size* is of no importance, and is often so altered, for constructive reasons, as greatly to disguise the nature of the mechanism without at all changing the motions belonging to it. This occurs specially where one element of a link is made so large as to include another, so that the actual outside form of the link is the form of one of its elements. As illustrations of this we may notice how such **expansion of elements** affects one or two of the simpler mechanisms whose motions we have already examined.

Fig. 174 shows an ordinary "lever-crank" chain, having four links *a, b, c* and *d*, connected by four turning pairs 1, 2,

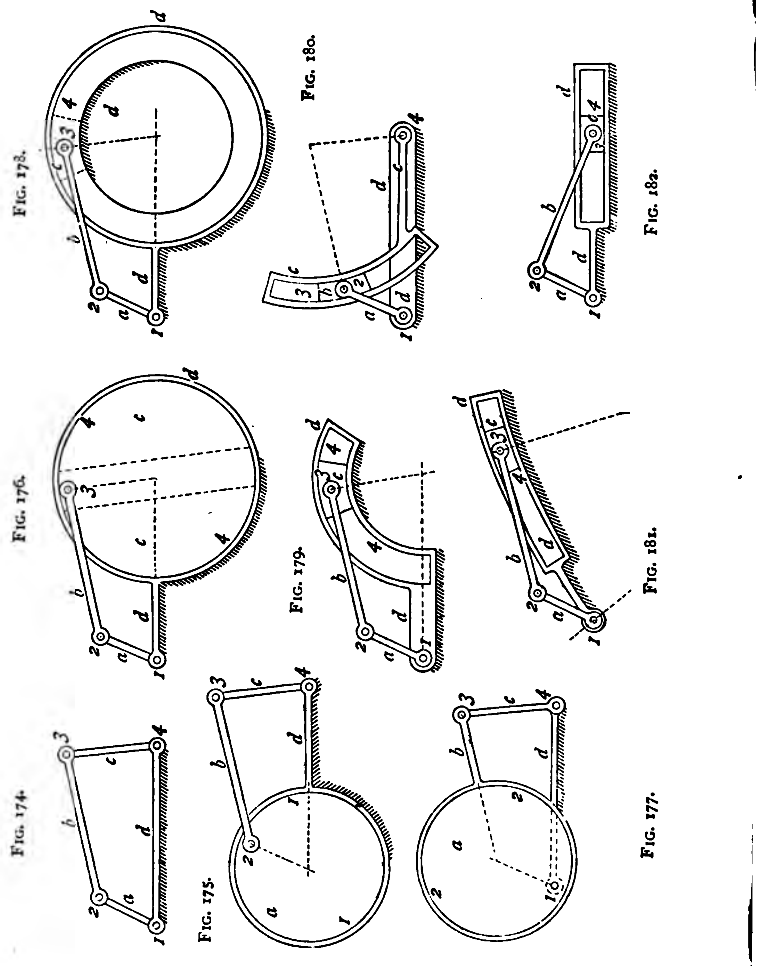


FIG. 174.

FIG. 175.

FIG. 176.

FIG. 177.

FIG. 178.

FIG. 179.

FIG. 180.

FIG. 181.

FIG. 182.

3 and 4. The same mechanism in precisely the same position is shown in Fig. 175, with only the difference that the pair 1 has been so much enlarged that the link *a* becomes a disc, whose periphery is one of its elements, and whose diameter is such as to include within it the pair 2. In Fig. 176 a similar change has been made with the pair 4 instead of the pair 1. In Fig. 177, again, the same change has been made with the pair 2, which now includes the pair 1. The link *a* in this case forms what is always called an *eccentric*, and comparing Figs. 177 and 174 we see at once the relation between the eccentric and the crank. Kinematically they are absolutely identical, their only difference lies in the *size* of the elements which they contain. Practically, of course, the great point of difference is that the eccentric allows the shaft at 1 to be carried through without a break on both sides of the mechanism, whereas with a crank the shaft has to be broken in the centre to allow for the swing of the connecting rod *b*.

But other changes, still more striking, can be made by altering not only the size, but what we may call the *extent* of the links. Thus for the link *c* in Fig. 176 we do not require the whole disc as drawn, it would be quite sufficient for us to use a narrow slice of it such as is shown in the dotted lines, having sufficient surface at its ends to constrain the motion. Or we may make *c* a complete ring (Fig. 178) instead of a disc, the part of *d* which forms its element of the pair 4 fitting inside as well as outside the ring. Treating *c* now again as we did in Fig. 176, it becomes a short sector or block, as shown in dotted lines. For this form of *c* the complete circles are no longer wanted in *d*, for *c* merely swings backwards and forwards without rotating. The mechanism, therefore, takes the form of Fig. 179, its motion still remaining absolutely identical with those of the

original mechanism, Fig. 174. By an exactly corresponding set of changes, which it is unnecessary to go over again at length, the mechanism might be made, without change in any of its motions, to take the form of Fig. 180, in which the link *b* becomes a block or sector, and one of the elements of *c* a curved slot. (To save space, the length of the sector in the figure is much shorter than would be necessary to allow *a* to turn completely round.)

In each of these last cases the link which has become in form a curved block, retains the same elements as before, altered only in diameter and angular extent. In each case, namely, these links contain two elements of turning pairs, and the centres of those elements,—which determine the distance apart of the pairs, or the real length of the links, —remain precisely as in Fig. 174. The distance 3·4 in that figure is therefore the actual length of the link *c* in Fig. 179, and the distance 2·3 the actual length of the link *b* in Fig. 180, these lengths being in no way altered by the external changes which we have made in the appearance of the mechanism.

The adoption of the block form of link shown in the last two figures has the practical convenience that it enables us to use very long links in a mechanism without necessarily making the mechanism itself very large. Thus in Fig. 181 the links *c* and *d* are made so long that the point 4 is inaccessible (at the join of the dotted lines), but the mechanism itself has become no larger in consequence, and the block *c* remains a link containing two turning elements, just as before. In this case the links *c* and *d* (*i.e.*, the lengths 3·4 and 1·4) are made equal, so that the centres of the pairs 1 and 3 lie on the same circle, having 4 for its centre. If now, everything else remaining unchanged, the point 4 be taken further and further away, the curvature of the slot

becomes flatter and flatter, until at last, when 4 is at an infinite distance, the slot becomes straight, the link *c* a straight instead of a curved block, and the mechanism becomes a slider-crank, Fig. 182, instead of a lever-crank. Here then we have the true relation between these important mechanisms. The slider-crank is derived from the other by making two of its links (*c* and *d*) equal, and at the same time infinitely long. The block of the slider crank, the reciprocating link, corresponds to the lever, the swinging link, of the lever crank, one of its elements unchanged, the other made infinite in radius.

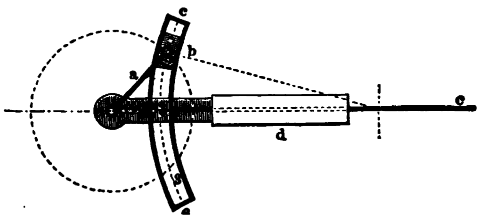


FIG. 183.

By now making in the slider-crank the same change as that made in Fig. 180, we obtain the form of mechanism shown in Fig. 183, in which the link *b*, without alteration of length, becomes a curved block instead of a long bar. And from this form it can be seen at once what will happen (Fig. 184) if *b* be made infinitely long, as well as *c* and *d*. The slider-crank changes into a form familiar as the driving mechanism of donkey pumps, and for other purposes. In reference to this mechanism it is often said that its working is equivalent to that with a connecting-rod infinitely long. It will be seen from our examination that this is true in a very literal sense. The length of a link in a mechanism we

take to be the distance between the axes of its elements. One of the elements of the link *b* has its axis at infinity, it is, therefore, not only equivalent to a link of infinite length, but actually *is* a link whose kinematic length (measured in exactly the same way as that of other links) is infinitely great. The mechanism shown in Fig. 185, and recognisable as a form of "trammel" for drawing ellipses, is identical with that of the last figure, and is illustrated merely to show the disguising effect of a few simple *external* changes. The nature of the pairing and the lengths of the links remain precisely as before. The form of Fig. 185 is,

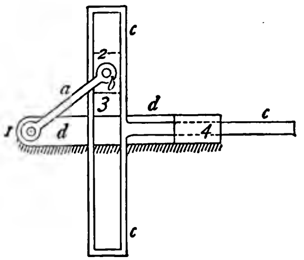


FIG. 184.

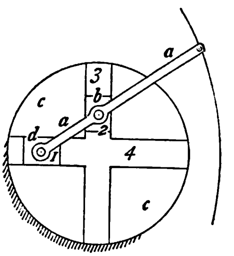


FIG. 185.

however, generally used when the link *c* is the fixed one, while with the former construction, *d* is generally made the fixed link. Letters and figures are the same in the two illustrations, and the student should satisfy himself by examination that they represent connections which are not only similar, but kinematically identical

The two following figures (Figs. 186 and 187) are taken from Reuleaux, and show to what an extraordinary extent the expansion of elements in a mechanism can alter its structure and appearance without changing its nature. The

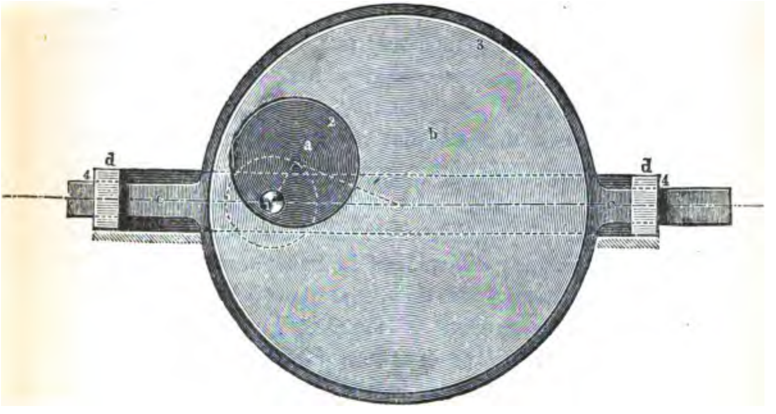


FIG. 186.

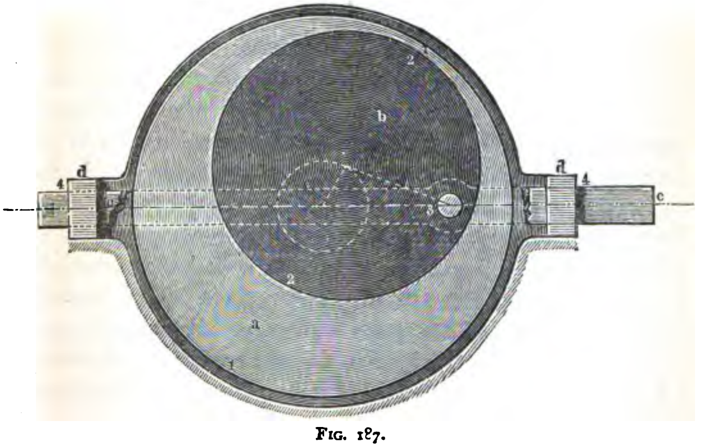


FIG. 187.

mechanism is an ordinary slider-crank in both cases, and the letters and figures on the links and pairs are the same as those used throughout this section. In Fig. 186, 1 is placed within 2, and 2 within 3, and in Fig. 187, 3 is placed within 2, and 2 within 1.

The varieties of form that not only can be, but have actually been obtained by expanding the elements of a mechanism variously, are innumerable. All, however, depend upon the principles here set down, and the student should not find any difficulty, if he has followed this section carefully, in tracing out their real nature. Reuleaux has illustrated a great many of them in his examination of rotary engines. Fresh examples will be found very frequently, if not every week, in the illustrated summaries of patents published by *Engineering* and *The Engineer.*

§ 53.—ALTERED MECHANISMS. REDUCTION OF LINKS.

It happens often, indeed in most cases, that one or more links of a mechanism are not *directly* utilised. Such links are wanted to constrain or transmit certain motions between certain other links, but their own actual motions are in no way required. The connecting rod of a steam engine is, for example, such a link. It transmits constrained motion from the piston to the crank. The piston is the driving link of the chain, and its to and fro motion is directly utilised in connection with the action of steam upon it. The crank is the driven link, and its rotary motion is essentially required for the sake of the machines which the engine has to drive. But the actual motion of the connecting rod is seldom utilised in any way. In certain important valve gears it is now used, and sometimes it is made use of for driving an

air pump, but these cases are quite exceptional. It has, therefore, often been thought, hastily, that such a link, only serving as a connection, would be better omitted, and that its omission (if only it could be made without destroying the constrainment of the mechanism) would be entirely advantageous. This omission of a link we may call the **reduction** of a mechanism, and a mechanism so treated will be said to be a **reduced mechanism.**

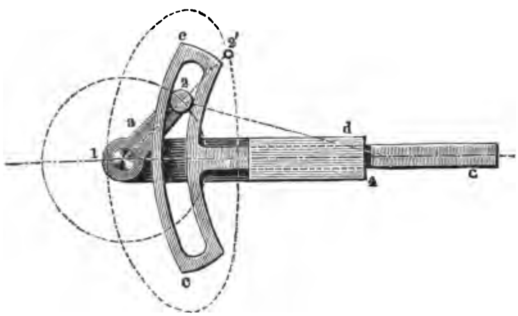


FIG. 188.

We have seen in § 10 that it was possible to constrain *any* plane motion, however complex, by a suitably formed higher pair of elements. It must, therefore, be quite possible to constrain the relative motions (to use the illustration of the last paragraph) of the crosshead and crank of a steam engine, by connecting them directly, omitting the connecting rod altogether. For this purpose it is only necessary to know the whole motion of the one body relatively to the other, and to construct a suitable higher pair of elements. One way in which this can be done is shown in Fig. 188. Here the link *b* of a slider-crank

chain being omitted, a pin is placed on the end of *a* and a slot made upon *c* of such form as to be the envelope for the various positions of the pin relatively to *d*. At first sight we seem to have altered nothing from Fig. 183 of the last section, but it takes very little examination to see that the omission of *b* has been accompanied by a serious practical drawback, namely the substitution of *line* contact for *surface* contact. This we know to be inevitable with the use of higher pairing (p. 57). If we could only suppose our links made with the extremest accuracy in dimensions, and of material so hard that its wear under ordinary forces was indefinitely small, there are very many cases in which the saving of a link would not be purchased too dearly by the substitution of higher for lower pairing. But in fact these conditions can never be attained. The wear in such a mechanism as that of Fig. 188, would be so great and so rapid that the motions of the links would speedily lose their required degree of constraint, and the machine would, in ordinary language, "knock itself to pieces."

There was, of course, no kinematic necessity for making the higher element upon *a* a circular pin; this has been done simply because it was most convenient to do so. Nor was there any necessity for placing it exactly where the crank pin formerly was; this also has been done merely for convenience' sake. The pin, for instance, might equally well have been placed at 2′ instead of at 2. But in that case the slot would have taken the form shown in the dotted line, which of course would have been very much more troublesome to make than the simple circular arc.

In Fig. 189 is shown a slider crank chain with the link *c*, the block, omitted. The links *b* and *d* are now connected by higher pairing, which has taken the form of a cylindrical pin on *b* working in a straight slot in *d*. As

before, the result is that by omitting a link we have had to replace the surface contact of two lower pairs by the line contact of a higher one, with the practical drawbacks already mentioned.

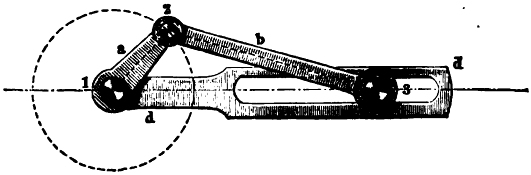


FIG. 189.

Fig. 190 shows another way in which *d* and *b* can be paired together if *c* be omitted. Here we have commenced by giving to *d*, for its element, the form of a straight bar, finding all the positions of this bar relatively to *b*, and constructing the envelope of these positions in the shape of curved profiles to projections placed upon the end of *b*.

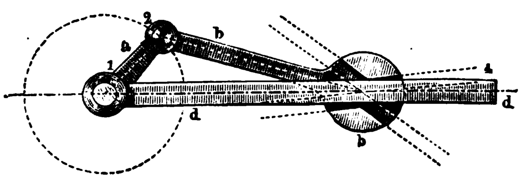


FIG. 190.

This form of higher pairing has been very frequently used in practice, apparently with imperfect recognition of the fact that it is incomplete in its constrainment, the smallest distance between the two curves being unavoidably greater than the breadth of the bar. At and near the ends of the stroke, therefore, the relative positions of *b* and *d* are not

absolutely fixed by the pairing, a defect which cannot be rectified without substituting some other form for that of the straight bar as the element of the pair belonging to *d*.

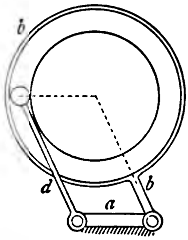


FIG. 191.

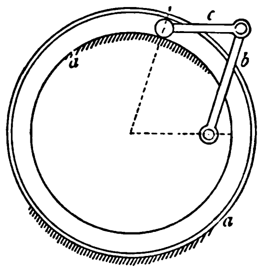


FIG. 192.

In Figs. 191 and 192 are shown two reduced forms of a linkwork parallelogram. In Fig. 191 the link *c* is omitted, in Fig. 192, the link *d*. In both cases the links formerly connected by the omitted link are now directly

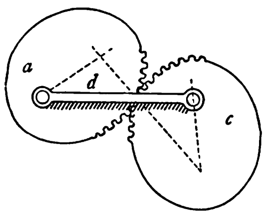
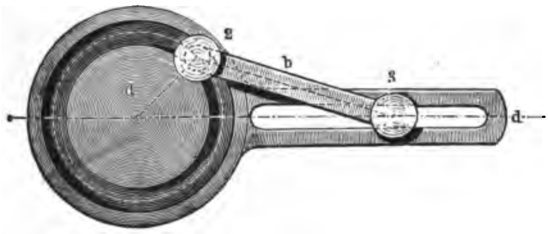


FIG. 193.

connected by higher pairing, and in both cases it has been possible to use for the higher pair a pin and a circular slot.

In Fig. 193 is shown a very different form of higher pairing, used in the mechanism already examined in Figs. 119 and 142, in which opposite links are equal but

anti-parallel. Here the link b is omitted, and the links a and c are paired by help of their centrodes, which are made into elliptic toothed wheels. In § 21, p. 150, we have already looked at the use of such wheels from another point of view.



F G. 194.

If it be desired to utilise only the motion of one link in a chain, all the others except the fixed link may be omitted, in which case the chain simply reduces itself to a pair of elements, necessarily a higher pair. Such a reduction,

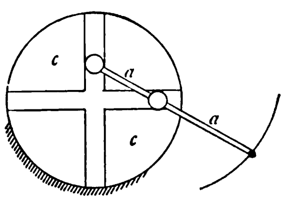


FIG. 195.

however, possesses, for engineering purposes, even greater drawbacks than the reductions already mentioned, and very seldom has counterbalancing advantages. Two cases of it are sketched in Figs. 194 and 195. The first of these shows a

slider crank from which the links *a* and *c* are omitted. The higher element on *b* takes the form of two circular pins,[1] and the corresponding element on *d* of two slots, one straight and one circular. There is, of course, only line contact throughout. Fig. 195 shows the converse case, when *b* and *d* are omitted, and *a* paired directly to *c*. The original form of the mechanism here reduced is shown in Fig. 185 of the last section, where its relation to the slider crank was discussed.

§ 54.—INCOMPLETE CONSTRAINMENT.

WE started in § 1 with the assumption that constrained motion was an absolute necessity in any combination that was to be used in a perfect machine. We have found, however, that there are many mechanisms which possess one or more unconstrained positions, and are to a corresponding extent unavailable or imperfect as machines. We shall in this section summarise the conditions under which such mechanisms are used.

A very common cause of want of constrainment is the existence of a **change-point**, already discussed in § 21, p. 147. We have there seen how a mechanism can be constrained at its change-point by compelling the centrodes corresponding to the required form of motion to roll upon one another, which effectually shuts out the possibility of any change. Another and more common method is to duplicate the mechanism with another, so placed that it is always in some completely constrained position when the first mechanism is passing its change-point. Perhaps the

[1] As to form and position of these pins, see remark in connection with Fig. 188.

most common illustration of this is sketched in Fig. 196, where a pair of parallel cranks *a* and *c*, connected by a coupling rod *b* (as in a locomotive), which would be unconstrained at two positions in each revolution, are made completely constrained by the addition of the duplicate cranks *a'* and *c'* (with the coupler *b'*) placed (say) at right angles to the original ones.

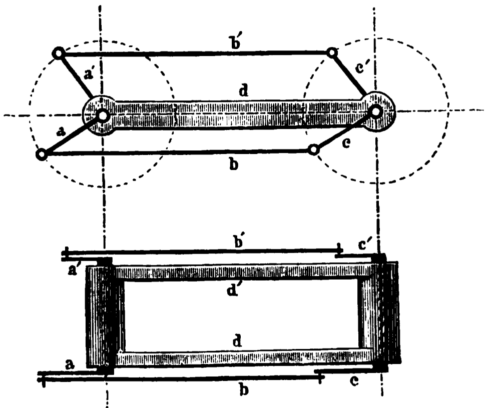


FIG. 196.

The existence of **dead points** in a mechanism is not to be confused with that of change-points. The change-point is inherent in the chain itself, and represents the possibility of change into some different chain or into a pair of elements. The dead point, on the other hand, is not inherent in the chain, or even in any particular mechanism formed from the chain, but depends on the particular link which is the driving link, and the particular way in which the driving force acts upon that link. Thus the ordinary slider-crank (Fig. 182), if it be used as the driving mechanism of a

steam engine, where c is the driving link, has a dead point at each end of the stroke of c. But if the same mechanism be used as a pump, where the crank is the driving link, and receive from any source a continuous rotary motion, there are no dead points.

The dead point, where it exists, may be passed by means of a duplication of the chain, such as has been described above as used for passing a change-point. The double slider-crank chain of Fig. 197 is a familiar illustration of

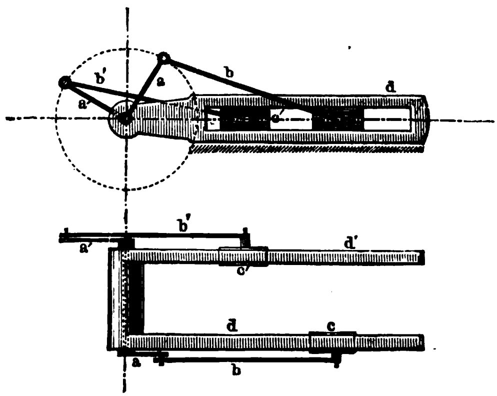


FIG. 197.

this. More often, however, the dead point is passed by help of a fly-wheel or other rotating mass, in which sufficient energy is stored up at other parts of the stroke. A constrainment of the former kind, effected by means of some addition to the mechanism, may be called a **chain-closure**, while constrainment by means of some special force or pressure, provided in connection with the masses of certain parts of the machine, may be called **force-closure** (see also § 51).

Force-closure is very frequently employed for completing the constraint of pairs of elements, when the form of one of them is left kinematically incomplete, as in the case of Figs. 198 and 199. In these cases it is generally the weight of one of the bodies which itself supplies the force necessary for constraint. In case of the occurrence of any disturbing force, this force takes the place of the resisting stresses which would, in a complete pair of elements, prevent change of motion.

It remains to mention, in this section, a curious case which occasionally occurs, in which a mechanism is employed whose motions, were they allowed to develop

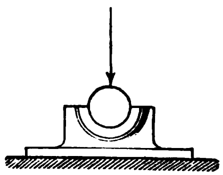


Fig. 198.

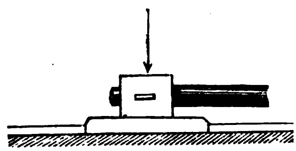


Fig. 199.

themselves, would be unconstrained, but in which only excessively small motions are permitted, and in which, after such motions have occurred, the mechanism is always brought back to precisely its starting position before it is made use of. Fig. 200 shows in outline a mechanism of this kind, which is used in some testing machines,[1] as a substitute for a train of levers having a very large "mechanical advantage." It consists of a lever, *b*, pivotted at 1 to one end of a fixed link, *a*, and loaded at its outer end. Its long arm is connected by *c* to a linkwork parallelogram,

[1] Those of Riehlé Brothers, of Philadelphia. They were exhibited, for instance, at the Philadelphia Exhibition of 1876.

d, *e*, *f*, *g*; the upper link of which carries at its outer end a small weight W_1 which balances a large resistance *W* on *b*. The link *g* is pivotted at 3 to the upper end of the fixed link *a*. The pair 2, connecting *c* to *d*, is placed so as not to be directly under 3, but some small distance to the right of it. The mechanism is clearly unconstrained, for either *b* or *g* could be fixed as well as *a*, and still all the other links could move, and this we know to be inconsistent with our original definition of constrainment in mechanisms.[1] Its want of constrainment comes out at once if the attempt be made to find its twenty-one virtual centres,—it will be

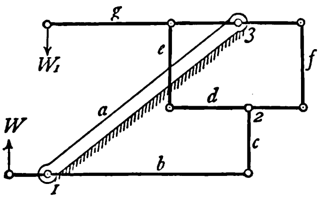


FIG. 200.

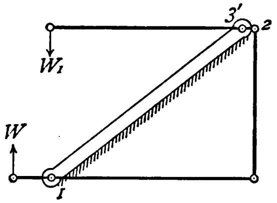


FIG. 201.

found that many of them are indeterminate. To constrain the mechanism it would be necessary to place another element upon *a* and to pair *d* to it at a point directly below the point 3. So far as the balance of *W* and W_1 is concerned the parallelogram then becomes superfluous, and the mechanism is statically equivalent to that of Fig. 201, which requires no further explanation. But in order that *W* may be as small as it is wished to be in the particular machine in question, the distance 3′2 must be excessively small—so small as to be constructively impracticable, owing to the absolutely necessary dimensions of the knife edges. The

[1] See § 11.

device of the parallelogram is therefore adopted in order to get 3 and 2 upon different links, so that the horizontal distance between them (the real length of the short arm of the lever *g*) may be made, with absolutely no constructive inconvenience, as small as is pleased. It was only one centimetre, for instance, in a 75-ton testing machine exhibited at Philadelphia in 1876. The purpose of the mechanism (as used in a testing machine) is to measure W by means of W_1. For this purpose it has to be assumed that the leverages of the mechanism, as constructed, are as determinate as those of the simple combination of Fig. 201, which it represents. This cannot be the case if even small changes in the position of the different links be permitted. In the machine, therefore, special arrangements are made by which both the links *b* and *g* can be placed accurately parallel and horizontal before the value of W_1 is read off. It would not be sufficient to have only one of the two links mentioned horizontal, because, as we have seen, that would not constrain the position of the other links. But if *both b* and *g* are brought into known positions, the positions of all the other links (and therefore the "mechanical advantage" of the mechanism as a whole) becomes determinate. The mechanism is therefore, as above described, moveable and unconstrained, but rendered available in a machine by making use of one of its positions only. How far this use of the mechanism is advisable, or within what limits its results are trustworthy, is a matter which has to be settled by practical experience.

§ 55.—THE PARALLELOGRAM.

THE simple linkwork parallelogram,—a four-link chain, with opposite links equal—has some special properties which deserve noting, both on account of their geometrical interest, and because they are so frequently utilised in practice. Such a parallelogram is shown in Fig. 202, its links lettered a, b, c and d. Let it be supposed that *only one point in it is fixed*, (viz., O, the join of a and d) instead of a whole link. Draw any line through O, as OBC, cutting

FIG. 202.

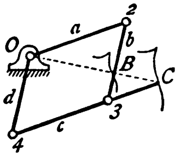
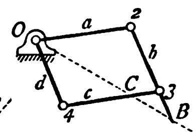


FIG. 203.



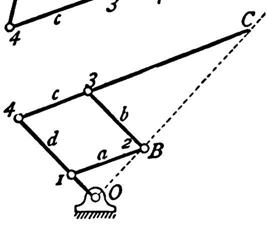


FIG. 204.

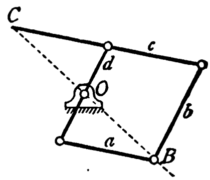


FIG. 205.

the two non-adjacent links in B and C. Then in whatever position the mechanism be placed, these three points will always lie upon one line. For by similarity of triangles, $\frac{3B}{4O} = \frac{3C}{4C}$, so that $3B = 4O \times \frac{3C}{4C}$, and B must therefore always occupy the same position on the link b. Further, the ratio $\frac{OC}{OB} = \frac{4C}{43}$ and is therefore constant for all posi-

tions of the mechanism. From this it follows that if B be made to trace any curve or line whatever, C will describe a precisely similar curve or line on a larger scale. By moving B upwards, the ratio of exaggeration can be increased to any extent. Or on the other hand by making OC *less* than OB (Fig. 203), the copy will be on a smaller scale than the original. The parallelogram finds numberless applications of this kind as a "pantagraph" or copying machine, for enlarging or reproducing maps or drawings.

It is not necessary that the fixed point O be at the join of two links, as in the foregoing cases; it may be taken at any point of any link, as O in Figs. 204 and 205. In this case one of the opposite pin centres, as 2 $(=B)$, becomes one of the two tracing points, the other lies at C upon the line BO. The ratio of exaggeration is $\frac{CO}{BO}$. In Fig. 205, this ratio is made equal to 1, so that the copy is a duplicate simply, of the same size as the original.

Professor Sylvester was the first to point out that the properties of the parallelogram just mentioned were not confined to points, such as C and B, lying in one line with the fixed point. In Fig. 206, *a b c d* are again four links of a parallelogram, of which the vertex O is fixed. The point P is *any* point on the link a, and the point P' a point on the link b so placed that the triangle $P'MQ$ is similar to the triangle MPN, the angles at P', M, and Q, being equal to those at M, P, and N, respectively. Then $\frac{PN}{NM} = \frac{MQ}{QP'}$ so that $\frac{PN}{MQ} = \frac{NM}{QP'}$ and therefore $\frac{PN}{NO} = \frac{OQ}{QP'}$. The angles PNO and OQP' are also equal. The triangles PNO and OQP' are therefore similar in all respects, and

$\frac{OP}{OP'} = \frac{ON}{QP'}$, which is a constant ratio. The ratio of the distances of P and P' from O is therefore the same for all positions of the mechanism. It can be readily shown, also, that the angle POP' remains constant for all positions of the mechanism. The points P' and P must therefore move in *similar* curves, so that one copies the motion of the other, but not only is the copy a different size from the original in the ratio $\frac{OP'}{OP}$,[1] but it is shifted from it round O by a certain definite angle POP'. An instrument for tracing curves in this fashion has been called by Professor Sylvester a *Plagiograph*, or *Skew Pantagraph*.

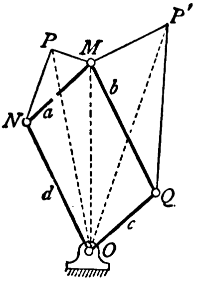


FIG. 206.

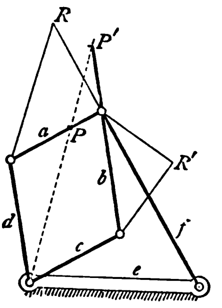


FIG. 207.

If we combine the parallelogram with other links to form a compound mechanism as in Fig. 207, we do not, of course, deprive it of any of its own properties, but merely determine the actual forms of the paths of its points. Thus P' still copies the motion of P, and R' of R, just as they

[1] Of course OP' may be made equal to OP, in which case $\frac{OP'}{OP} = 1$, and the copy is a duplicate of the original.

would if the point *O* only were fixed. But out of this mechanism we can construct two four-link mechanisms, each containing two links of the parallelogram, viz. *a*, *d*, *e*, *f*, and *b*, *c*, *e*, *f*. These two mechanisms will evidently have the property that for every point (as *P* or *R*) on the link *a* of the one, there can be found one point (as *P'* or *R'*) on the link *b* of the other such that the path of the latter point shall be similar to that of the former. This curious property was first pointed out by Mr. Kempe.

One of the principal uses to which the parallelogram is put in practice is the copying of the approximately straight line drawn by one point of a parallel motion. This will be considered more fully in the next section. Other properties of the parallelogram will also be noticed in § 57.

§ 56.—PARALLEL MOTIONS.

UNDER the somewhat inappropriate name of "parallel motions" are included in this country certain mechanisms possessing the characteristic that one or more points in them, not directly guided by sliding pairs, move approximately or accurately in straight lines. These mechanisms may be divided into three classes: (1.) those in which the straight line is merely an extended copy of a line constrained by a sliding pair somewhere else in the machine; (2.) those in which the mechanism contains a sliding pair, but without copying its motion, and (3.) those in which all the links of the mechanism are connected by pin-joints, that is by turning pairs. This last class is again subdivided into mechanisms in which the so-called straight line is merely an approximation, and those in which it is mathematically

exact, this latter class being or very modern origin.[1] We shall look at these different mechanisms in the order in which they have been mentioned.

The mechanism belonging to the first class which is most commonly used as a parallel motion is a slider crank in which the connecting rod is made equal in length to the crank. We have already examined this mechanism (§§ 21 and 42, pp. 146 and 318) and seen that the centrodes of the links *b* and *d* are a pair of circles, one of which is twice the diameter of the other. Any point, therefore, on the link *b*, as *M* (Fig. 208), which is at a distance 23 from the point 2, will describe a straight line, passing through the point 1, relatively to *d*. In order that this line may reach its maximum possible length of four times the crank radius, the sliding pair 4 must have a travel equal to twice the crank radius.[2] This would have many inconveniences in practice, so that only the central part of the link is used, and this allows of the employment of a sliding pair with comparatively very short travel, as shown in the figure. This parallel motion is generally known in this country as Scott Russell's.

Another parallel motion, but one not so well known, is based on the mechanism of Fig. 185, § 52, which is, as we have seen, a slider crank in which three links (*b*, *c* and *d*) are made infinitely long. In this case (Fig. 209) it can be seen at once that the point O_{ac} is the virtual centre for the links

[1] On this part of the subject, see particularly Mr. A. B. Kempe's paper "On a General Method of Obtaining Exact Rectilinear Motion by Linkwork" in the *Proc. R. S.*, 1875, as well as other papers cited by him in his lectures, *How to Draw a Straight Line* (Macmillan, 1877), this last a most interesting elementary statement of the matter. To Mr. Kempe's work I have been very much indebted in writing the latter part of this section.

[2] It is here supposed that the link is not required to rotate, but may merely swing backwards and forwards.

a and *c*, and it may be left to the student to prove that the centrodes of these links are again circles as in the last case, the *radius* of the centrode of *c*, and the *diameter* of that of *a*, being equal to the distance $O_{ac}S$. Hence, exactly as in the last case, any point of *a* which lies upon its centrode will describe a straight line (passing through the point *S*) relatively to *c*. The two sliding pairs may have, as in the last case, a travel comparatively much smaller than the stroke of the describing point *M*. The two sliding pairs may be at right angles to each other (as in Fig. 185), or at any other angle to each other, as in the figure here given.

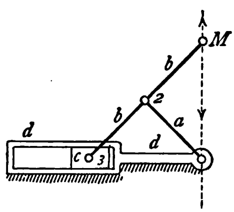


FIG. 208.

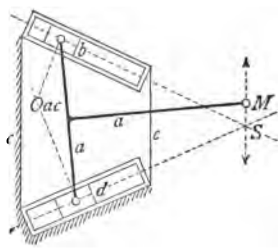


FIG. 209.

In both these cases it has been mentioned that *any* point upon the circular centrode might be used as the guided one. If required, therefore, several might be used simultaneously, and then the one (Fig. 208), or the two (Fig. 209) auxiliary sliding pairs might be made to constrain any number of points to move in straight lines. In each case all the straight lines will pass through one point.

Fig. 210 is an example of the second class of parallel motions enumerated above, in which a sliding pair is still employed, but by which only an approximation to a straight line is obtained. In order that the centrodes of two bodies rela-

tively to each other may be the two circles whose properties we have just been able to utilise, it is necessary and sufficient that any two points of one should describe non-parallel straight lines relatively to the other. It follows (in consequence of the rolling of the centrodes) that all other points of the first body should describe ellipses relatively to the second. The relative motion of the two bodies would be just as completely constrained by making two points move in two of these ellipses as in the two straight lines, but this would not, of course, be practically so convenient. There is no difficulty, however, in finding circular arcs which very closely coincide with certain portions of these ellipses, and in getting rid, by their use, of one or both of the sliding pairs without causing the

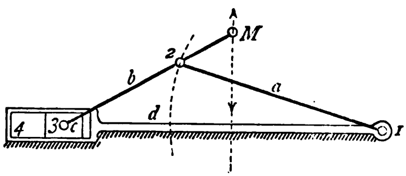


FIG. 210.

described line to vary much from accurate straightness. In Fig. 210 one sliding pair is retained, the path of 3 is therefore made accurately straight, but is (as explained in connection with Fig. 208) of very limited extent. The point *M* is the describing point, but the point 2 is not taken in the centre of 3 *M*, so that its path (if *M* moves accurately in a straight line) is an ellipse. The actual motion of 2 is, however, a circular arc, with centre at 1, as constrained by the link *a*. If this circular arc sufficiently nearly coincides with the ellipse instead of which it is used, the path of *M* may be assumed, for most purposes of practical engineering, to be a straight line. The point 1 for any given position of 2 can

be best determined by finding the highest, lowest, and middle positions occupied by 2 if *M* does move in a straight line, and then using for 1 the centre of a circle drawn through these three points. The swing of the link *b* on each side of the centre line should not exceed 40°, and the approximation is of course closer if the angle be smaller. As compared with the exact motion for which this is a substitute, it has the constructive advantage that the path of *M* does not pass through the point 1.

If a body move so that one line in it passes always through one point, and one point in that line describes a straight line, its other points describe curves of a high order known as conchoids. These curves have, under suitable conditions, portions which are very nearly circular. If therefore, with suitably chosen points, we cause a line to move so that it always passes through one point, and cause one point in that line to describe a circle closely coinciding with a conchoidal arc, some other point in the line will describe approximately a straight line.

Such a motion is obtained by the use of the inverted slider-crank mechanism (link *b* fixed, as in the oscillating engine) of Fig. 211. Here the point 1 in the link *d* is constrained by the link *a* to turn always about 2, in an arc approximately coinciding with the conchoid, and the sliding pair at 4 compels the line 1 *M* of the same link to pass always through the point 3. The paths of a few points of *d* are shown in dotted lines. The point *M* has a path which for a short distance may be taken to represent a straight line.

With a moderate angular swing this mechanism gives a very good approximation to a straight line. The best position for the centre 2 can be found, for a given point 3 and a given length *M* 1 and path of *M*, as before, viz., by finding the positions of 1 for highest, lowest, and middle positions

of *M*, and taking for 2 the centre of a circle passing through the three points so found.

The third class of parallel motions, those in which only pin joints are used, may be first illustrated by two forms which are used as approximations to the exact parallel motion obtained by the help of sliding pairs; they are shown in Figs. 212 and 213. Thus for instance *Roberts'* motion, shown in Fig. 212, is derived from the trammel motion of Fig. 209. The straight paths of the two end points of the link *a* are replaced by circular arcs, approximations, in reality, to

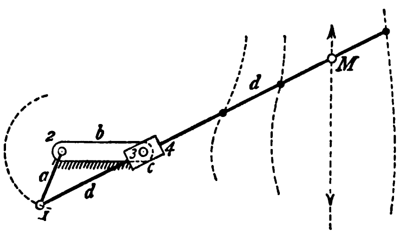


FIG. 211.

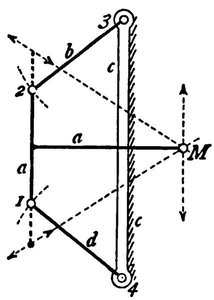


FIG. 212.

the elliptical paths of other points of that link, the links *b* and *d* being connected to *c* by pins instead of by sliding blocks. The point *M* describes a good approximation to a straight line for a certain part of its path.

Most commonly the point *M* is made to lie on the line 34, so that its path coincides with the axis of the link *c*. In this case the lengths 23, 2 *M*, *M* 1, and 14 are all equal, and should be *not less than* 0·42 of the length of the link *c*,[1] and as much greater as possible. The length of *a* will obviously be equal to

[1] Rankine, *Machinery and Millwork*, chap. v. sect. 5.

half that of *c*. If it be inconvenient to make the path of *M* coincide with the line 34, it may be placed outside it, as in the figure. In that case the best points for 3 and 4 for any assumed triangle 2 *M* 1 will be found as in the former cases by finding the three positions of the points 2 and 1 for the ends and middle of the travel of *M*, assuming that travel to be accurately straight, and then taking 3 and 4 as the centres of circles passing through each set of three points.

Fig. 213 shows a linkwork parallel motion which gives an approximation to the already only approximate rectilinear motion obtained in Fig. 210. The infinite links *c* and *d* of

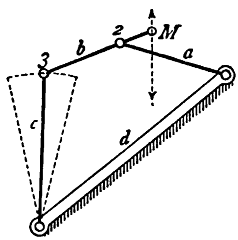


FIG. 213.

the slider-crank are replaced by ordinary links, and the straight path of the point 3 by an arc of a circle. So long as the length of *c* is not less than the whole travel of *M* which is utilised as straight, the approximation given by this mechanism is sufficiently good for most practical purposes. The point 1 must be determined in the way described in connection with Fig. 210.

By using for 3 not the point originally moving in a straight line, but some other point of *b* whose proper path is an ellipse—and so substituting the circular arc for an elliptic one (as already with the point 2) instead of for a straight

line—we can obtain other modifications of the mechanisms, which may sometimes be convenient.

Fig. 214 shows the ordinary *Watt* motion, the best known and most often used of all the approximate parallel motions. In the most common and best form of this mechanism the links *b* and *d* are equal, and the describing point *M* is in the middle of the link *a*; the length of *a* is made about equal to the intended stroke of *M*; in their mid-positions *b* and *d* are parallel and lie at right angles to the path of *M*, and the points 1 and 2 deviate to right and left of that path by equal amounts at the middle and ends of their swing.[1]

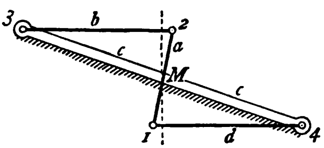


FIG. 214.

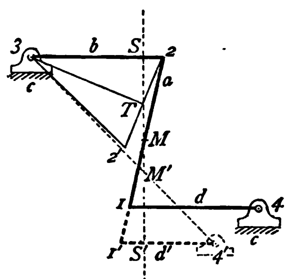


FIG. 215.

In any given case it is advisable to obtain as many of these conditions as possible, even if all cannot be simultaneously secured.

The following are some of the principal constructions connected with the Watt parallel motion. Let there be given (Fig. 215) the path of the point *M* and its mid-position, and the axis of the link *b* in its mid-position with its centre 3. It is required to find the point 2 which fixes the length

[1] See as to this and the constructions following, Rankine, *Machinery and Millwork*, chap. v. sect. 5.

of b, and the distance $2\ M$. The condition to be fulfilled by 2 is that its middle and end positions should be to right and left of the path of M by equal amounts. Make ST equal to one quarter of the stroke of M, and draw $T2$ at right angles to $3\ T$. The point 2 will then be in the required position. 32 will be the length of b, $2\ M$ will be a part of the link a in its mid-position, and $2'$ will be the position of 2 when at the end of its stroke ($2'\ T = 2\ T$). If $M\ 1 = 2\ M$, then $d = b$ and the point 4 can be found at once. But if 1 be at any point $1'$, so that $M1'$ is not equal to $2\ M$, then the best result (to make the point 1 fulfil the conditions above prescribed for 2) will be obtained by setting off $S'M' = SM$, and drawing $3\ M'$ to get the point $4'$, as shown in dotted lines in the figure. The link d then becomes d', with a length $1'4'$ and a centre at the point $4'$.

The complete curve traced by the describing point in the Watt motion is in form a distorted figure-of-eight, called a *lemniscoid*. The part actually used for a straight line is in reality wavy, and has five points which actually do lie upon one straight line. In the best forms of the mechanism three of these five points coalesce in the centre point.

As used in beam engines, the Watt parallel motion is generally combined with a copying mechanism, in the shape of a parallelogram (see § 55), for increasing the length of the line in a way not involving so much weight and space as would the enlargement of the parallel motion itself. This is shown in Fig. 216. The parallel motion proper consists of the four links a, b, d and the fixed link c. M is the guided point, as before. The link b is generally the main beam of the engine, and it would be very inconvenient to connect a to the end of the beam and provide by a huge link d' the direct parallel motion for the point E, above the piston rod. Some point such as M therefore, which has

much smaller stroke, is directly guided, and additional links *e* and *f* are added to form a parallelogram such that *E*, *M* and *O* lie all upon one line. (In practice the point *E* would be first fixed, and *M* found from it.) It was shown in § 55 that in this case *E* would copy the motion of *M*, and the motion of *M* having already been made (approximately) rectilinear the motion of *E* will be (approximately) rectilinear also.

In most cases the point 2 is taken at half the beam radius, and $\frac{OE}{OM} = 2$. The points 5 and 1 then coincide, while the points 4 and *E* and the axes of the links *d* and *f* appear to coincide when in their mid-position.

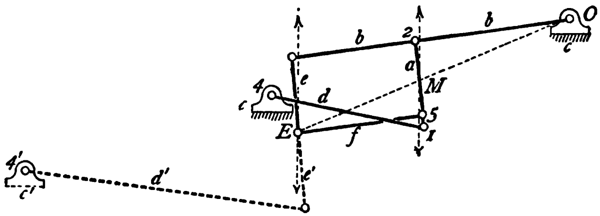


FIG. 216.

The motion of the point *M* is generally utilised for one of the pump rods of the engine. By adding other parallel links to the parallelogram other points can be found also moving in straight lines, which may be, and often are, utilised for other pump rods on the engine.

Perhaps the most modern of approximate parallel motions is that of Professor Tchebicheff, of St. Petersburg, of which there was a small example in the Vienna Exhibition of 1873.[1] This motion is shown in Fig. 217 and gives a very

[1] *Engineering*, vol. xvi. p. 284.

good approximation to a straight line. The links *b* and *d* are equal, and may be made each about 1·3 times the length of *c*. The length of *a* may be 0·4 of the length of *c*. The describing point *M* is in the middle of *a*. The travel of *M* may be anything less than the distance 34. The points 3 and 4 should be found as already described for the Roberts' parallel motion.

The first of the *exact* linkwork parallel motions was invented as recently as 1864 by M. Peaucellier, a French engineer officer. We have already once or twice (Fig. 128) used it in problems, but without examining its special theory or properties, which we shall now proceed to do.

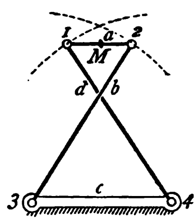


FIG. 217.

The Peaucellier parallel motion is a compound mechanism having eight[1] links. Of these eight, four (*a*, *b*, *c*, and *d*) are equal, and form a rhombus in all positions (see Figs. 218 and 219). The others are equal in pairs, viz., *e* and *f* are equal, of any length that will permit them to be

[1] It is usually called a "*seven-bar*" mechanism, the eighth or fixed link not being counted, and the same nomenclature has been used for the other exact parallel motions. This is, I think, to be regretted, for the fixed link is just as much, or as little, a bar as any of the other links.

jointed to each other at one end (P), and to opposite angles of the rhombus (S and T) at the other. The remaining links, h and g, are also equal; they are jointed to each other at Q, to the common point of e and f (P), and to a third angle of the rhombus (N), in the fashion shown in the figures. If the link h be now fixed, the remaining angle of the rhombus M will move accurately in a straight line at right angles to the axis PQ of the fixed link.

FIG. 218.

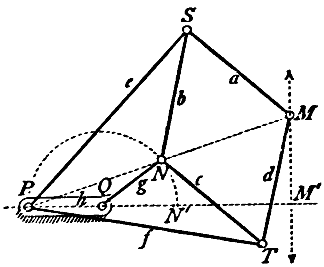
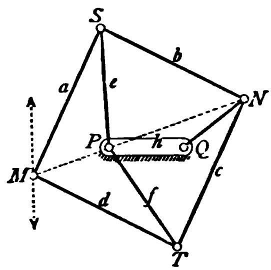


FIG. 219.



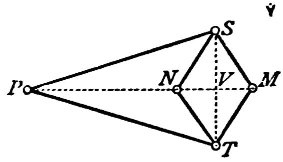


FIG. 220.

The six links first mentioned, a to f, form together what is called a *Peaucellier cell.* Of the other two links one is the fixed link, and the other (g) swings so that its end point N moves in a circle passing through P.

Considering in the first place only the Peaucellier cell (Fig. 220), we notice that (from the given conditions as to equality and symmetry of its links) the three points P, N,

and M must always lie on one straight line. If now we call the mid-point of the rhombus V, we have

$$\begin{aligned} PS^2 &= PV^2 + VS^2, \text{ and} \\ SM^2 &= MV^2 + VS^2, \text{ from which} \\ PS^2 - SM^2 &= PV^2 - MV^2 \\ &= (PV - MV)\,(PV + MV) \\ &= PN.\,PM \end{aligned}$$

As PS and SM are constants for any given mechanism, the product $PN.PM$ must be constant also, *whatever the position of the mechanism.*

Going back now to Fig. 218, which shows the complete mechanism, and supposing N to be at N', the point opposite P, and M therefore to be in some position M' in line with PN', then

$$PN.\,PM = PN'.\,PM', \text{ and}$$
$$\frac{PN}{PN'} = \frac{PM'}{PM}.$$

The triangles PNN' and $PM'M$ must therefore be similar, as the angle at P is common to both of them. But the points N', N and P lie, by construction, upon a circle of which PN' is a diameter, the angle PNN' is therefore a right angle, as being the angle in a semicircle. The angle $MM'P$ is therefore a right angle also. But this will be true whatever the position of the mechanism, that is, for any possible position of the point M. Hence *the point M must move so as always to lie upon—in other words so as to describe—a straight line at right angles to PQ, the axis of the fixed link.*[1]

[1] In this case the straight line is described as the *inverse* of the circle described by N. If N described some other curve than a circle, M would describe the inverse of that curve, which would, of course, not be a straight line. It was from this property of the Peaucellier cell

In concluding this section we may now go on to consider the more general cases of exact linkwork parallel motions which have recently been discovered. Although several other mathematicians, notably Professor Sylvester, have worked at these problems, we are indebted to Mr. Kempe (whose papers have been cited above) for the most complete and general investigation of them, of which a few only of the leading features are given in the following paragraphs.

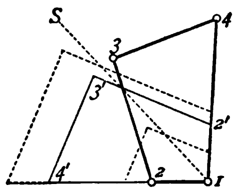


FIG. 221.

Let 1234 (Fig. 221) be any four-link mechanism of the lever-crank type. If the angle 214 be bisected in 1*S*, and a *similar* mechanism 12′3′4′ be placed symmetrically to 1*S* as an axis, the second mechanism is said to be the **image** of the first. The image need not be the same size as the original mechanism, but may be reduced or enlarged in any ratio, as shown in dotted lines. It remains an image so long as its links are parallel to those of the original equal image 12′3′4′.

that its value as a parallel motion was originally discovered. Mr. Kempe has shown, however, that the Peaucellier motion may really be taken as a very special case of the more general parallel motions now to be described. It may be further noted that if the links *g* and *h* be *not* equal, *M* will describe the arc of a circle, and the mechanism may be utilised in this form for describing accurately circular arcs of very large radius.

In Fig. 222 *abcd* is an ordinary four-link mechanism, of which a is the fixed link. Conjoined with this is a second mechanism 12′3′4′ ($a'b'c'd'$), which is a reduced image of the first. The axes of the links a and d' are made to coincide, and also those of d and a'. The ratio in which the second mechanism copies the first, *i.e.* the ratio

$$\frac{a'}{a}=\frac{b'}{b}=\frac{c'}{c}=\frac{d'}{d} \text{ we may call } =K.$$

It can be shown that in all positions of this compound mechanism its one part remains an image of the other. The fixed link of the mechanism carries three elements. One is paired with the compound link da', each of the others is paired with a simple link, here b and c'. If now any point (as P) be taken on one of these two links (as b), it can be shown that it is always possible to find a corresponding point (as P') on the other (as c'), the distance of which from P, measured parallel to the axis of the fixed link, shall be the same for all positions of the mechanism. It is on this property of this type of compound mechanism that its usefulness as a parallel motion depends. To utilise it, it is necessary to choose a particular point P, and to obtain the most convenient form of mechanism we must also choose a particular value for the ratio K. We get what is perhaps the most general form of exact parallel motion by giving K *any* value, but making

$$2\,\mathrm{P} = \frac{(a-Kd)\ ab}{(a^2+b^2)-(c^2+d^2)}$$

With these proportions the constant distance NN' becomes equal to half of the distance $24'$. If then we add to our six links two others, PM and MP', the one equal to $P2$ and the other to $4'\,P'$, the points P and P' will remain always

the vertices of two isosceles triangles whose bases lie in the line 14′, and the point M must move exactly along that line.

An immense number of modifications of this mechanism can be devised, but perhaps the most convenient is that shown in Fig. 223, which moves with great freedom, and has a relatively enormous stroke for its describing point. In this mechanism the value of K is made equal to $\frac{d}{a}$, and the link b equal to the link c, while

$$2\,P = \frac{(a - Kd)\;ab}{(a^2 + b^2) - (c^2 + d^2)} = \frac{a^2b - d^2b}{a^2 - d^2} = b.$$

These proportions give us a mechanism in which $a' = d$, $b = c$, $b' = c'$, P coincides with 3, P' with 3′, and 2′ with 4. Of

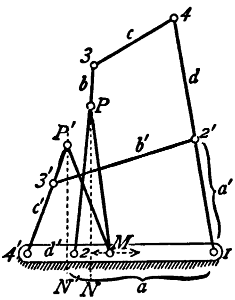


FIG. 222.

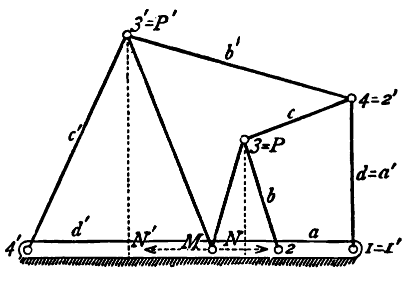


FIG. 223.

course from a constructive point of view it is awkward to allow the point M to pass the points 4′, 2. and 1, but notwithstanding this it is very possible that there may be some cases in which the other advantages of this mechanism may cause it to be practically used, cases, namely, where a slider may be inadmissible, and where an ordinary approximate motion would be too unwieldy for the long stroke required.

We shall consider only one more form of exact linkwork parallel motion, namely that of Mr. Hart,[1] which is geometrically most notable because it contains only six links instead of eight, but which is not perhaps of so much practical interest as those already described, because the dimensions of the mechanism are very large in proportion to the length of the line described. Fig. 224 shows an anti-parallelogram, 1234, cut by any line PM parallel to 13 or 24. Such a line cuts the axes of the links in four points,

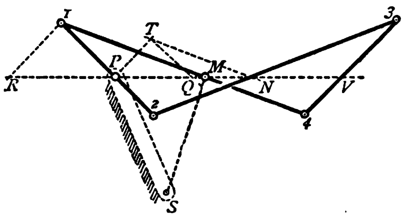


FIG. 224.

$PNMV$, which divide the links in a constant ratio, that is, $\frac{1P}{P2}=\frac{1M}{M4}=\frac{3N}{N2}=\frac{3V}{V4}$. The four points $PNMV$, therefore, will remain always in one line, however the mechanism moves. If we turn $P2N$ round to PTN, and draw TQ parallel to 12 and $1R$ parallel to TP, it can easily be seen that in $M1RP$, and $NTPQ$ we have halves of two Peaucellier cells. From the proof already given we know, therefore, that each of the products $NP.\ NQ$ and $MP.\ MR$ will be constant for all positions of the mechanism; therefore the joint product $NP.NQ \times MP.MR$, must be constant also. But by symmetry $\frac{MP}{MR} = \frac{NQ}{NP}$, so that of the four

[1] *Cambridge Messenger of Mathematics*, 1875, vol. iv. p. 82, &c.

quantities just given, $MP.NP = MR.NQ$, from which it follows that

$$PM.\ PN = \text{constant}$$

here, as in the Peaucellier cell, for all positions of the mechanism. (Similarly it can be shown that $VM.VN.$, $PN.NV.$, and $PM.MV$, are all products which remain constant for all positions of the mechanism.) We can therefore convert this anti-parallelogram into an exact parallel motion by the addition of two equal links, exactly as in the former case. Thus PS and SM may be added, and PS fixed. The point N will then describe a straight line at right angles to PS.

§ 57.—PARALLEL MOTIONS. (*Continued.*)

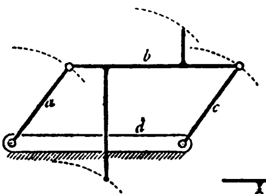
The name "parallel motion" is in this country so firmly connected with the straight-line mechanisms discussed in the last section, that it would seem pedantic to deny it to them. It is nevertheless a somewhat unsuitable name for them, and describes much better another class of mechanisms which we shall now examine, and in which one or more links are constrained to move always parallel to themselves. For these mechanisms no special name seems to have been proposed, and we shall not attempt to supply the deficiency.

The simplest of these mechanisms (disregarding, of course, those in which the desired motion is obtained by the use of sliding pairs) is the parallelogram itself (Fig. 225), in which if any link be fixed the opposite link has a motion of translation only, all its points moving with equal velocity in circular paths of equal diameter. This applies not only to points along the axis of the link, but to any points whatever

connected with it. A few such paths are sketched in the figure. In its simplest form the parallelogram is used in the ordinary "parallel ruler," where only the parallelism of *b* and *d* is utilised, motions of points being left out of the question.

Fig. 226 shows a common application of this in what is known as the Roberval balance. The parallelogram is doubled by the addition of one link, *c'*, and the links *c* and *c'* remain always parallel to each other, and at the same distance from *a*. The scale tables, therefore, connected with

FIG. 225.



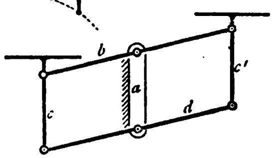


FIG. 226.

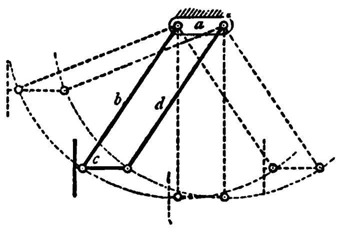


FIG. 227.

them, remain always horizontal. The often-proposed type of "feathering paddle-wheel" shown in Fig. 227, utilises the same mechanism in the same fashion. It should be mentioned, perhaps, that this motion is quite different from that really required in a feathering paddle-wheel, and that it would be quite useless for any such purpose.

If two Watt motions, having equal radius rods, be combined, as in Figs. 228 and 229—the mechanisms having been proportioned in the way given on p. 425—the distance

between the two describing points M and M' would remain constant if those points described accurately straight lines, and will in practice change so extremely little that it may be assumed to be constant. They may therefore be connected

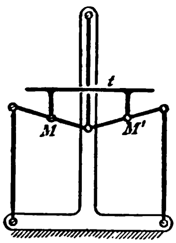


FIG. 228.

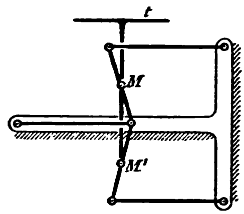


FIG. 229.

by a link t, and all points in that link will move in parallel and (approximately) straight lines. Such a mechanism would form a very easily working linkwork carriage for a straight moving table where the use of slides was objectionable.

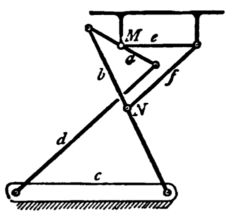


FIG. 230.

Fig. 230 shows another approximate motion of the same type (but with one link less) proposed by Mr. Kempe, and based upon the Tchebicheff parallel motion already described (see Fig. 217). The length of the link e, to which the table is connected, should be half that of the fixed link c, and the

length of *f* one-half that of *b* or *d*. The point *N* lies midway along the link *b*.

In conclusion, three of Mr. Kempe's mechanisms may be given, which, although more complex than those just looked at, give motions which are exact instead of only approximate. The mechanism *a b c d* (Fig. 231) is a "kite" (or four-link mechanism in which adjacent links are equal, in which the long links *c* and *d* are made twice the length of the short ones, *a* and *b*. With it is compounded another kite, *a′ b′ c′ d′*, exactly half its size, in such a way that *d′* coincides with *a*, and *a′* lies along *d*. It can readily be shown that in such a mechanism the line joining *M* and *M′* is

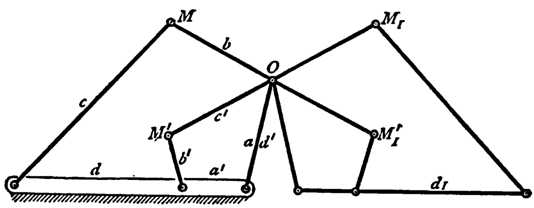


FIG. 231.

always perpendicular to the axis of the link *d*. Let now the links *b* and *c′* be extended through *O* to M'_1 and M_1 respectively, and let a new double kite, exactly equal and similar to the first, be constructed on these extended links. The points M_1 and M'_1 must always lie on a line parallel to *MM′*, and the new link d_1 must therefore always remain parallel to the original link *d*. Further, as *d* is fixed, the symmetry of the mechanism constrains d_1 to remain always not only parallel to, but *in line with*, *d*. Thus any table, or other body, attached to d_1 will have a simple motion of translation, all its points moving in parallel straight lines, not only approximately, but exactly.

In Fig. 232 is given a modification of this mechanism in which the link d_1 is constrained to move at right angles to the fixed link d, instead of parallel to it. The mechanism consists of the same pair of double kites as before, but differently connected together.

Fig. 233, lastly, shows how this motion can be applied to an ordinary double "parallel ruler," to constrain not only parallelism of position but straight-line motion. Three of the links of the last figure are omitted, and their places taken by three links of the double parallelogram. Two kites remain, one large and one small, compounded externally with

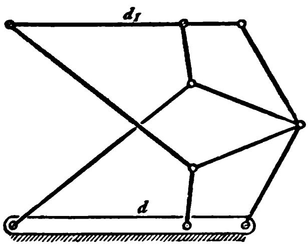


FIG. 232.

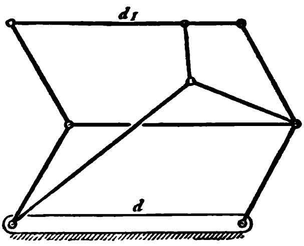


FIG. 233.

each other and with the two outer links of the parallel motion. The motion of d_1 relatively to d is exactly the same as in the last case.

It is perhaps well to point out that the practical applications of many of the mechanisms described in this section are very limited in number. It was long ago pointed out by Dr. Pole[1] that the greater loss by friction in guides than in the pins of a parallel motion was after all not a quantity sensibly affecting the economical working of a steam engine, and the same thing is true for other machines. The making

[1] See *Proc. Inst. Civil Engineers*, vol. ii. (1843) p. 69.

of truly plane surfaces is now, moreover, comparatively easy and inexpensive, and it is generally more easy to ensure that the motions of slides shall be unaltered by the wear of these surfaces than that the motions of points in linkwork shall be unaltered by the smaller, but less regular, wear in the pins and eyes. Hence in ordinary machinery, where large forces come into action, and where therefore much wear accompanies the motion, it is not probable that linkwork connections will again take the place of slides for the constrainment of parallel motions. In the beam engine, however, the Watt motion still holds its own, and seems likely to continue to do so, and there are probably not a few cases of light instruments, or experimental apparatus, where the effects of wear are small enough to be left out of account, and where the "sweetness" of the linkwork motions would rightly cause them to be preferred to slides.

§ 58.—ORDER OF MECHANISMS; CHAINS WITH LOWER PAIRING (PINS AND SLIDES).

ALL the mechanisms hitherto examined or used for illustration, which have contained only "lower" pairs (namely pin joints or slides)—except the doubled double-kite of the last section—have belonged to what may be called the **first order**. In all of them (with the exception named) it was possible, if the positions of any two links were given, to find the positions of all the other links by mere straight-line and circle constructions. Or putting it otherwise, we may say that it was possible, if the mechanism were given in any one position, to find by such constructions all its other possible positions. In all of them, also, it was possible, by equally simple constructions, and by the use of the theorem of the

three virtual centres (see p. 73), to find the virtual centre of every link relatively to every other. Although both these conditions are fulfilled by the great majority of mechanisms with which we have to deal practically, which, therefore, belong to the first order, there are some mechanisms—and these not unimportant—which cannot be dealt with in this fashion, and which, therefore, belong to higher orders. Without attempting here to classify completely such mechanisms into different orders—for probably all those which are of much practical importance belong to two orders—we shall simply examine two or three of them in order to show their characteristics, and the way in which they require to be handled.

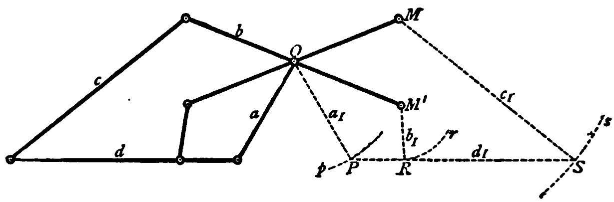


FIG. 234.

As a simple, although unusual example, let us take first Mr. Kempe's compound kite chain (Fig. 234), some of whose properties have been noticed in the last section. Suppose the links c and d to be given in any of their possible relative positions, and the lengths of all the other links to be given also. As the mechanism is completely constrained we know that the position of all these links must be completely determined by the given positions of c and d, and it is desired to draw the whole mechanism in the position thus determined. The first double kite can obviously be drawn at once, and if we knew the properties of this particular mechanism, namely,

that the two double kites must be symmetrical, and that d_1 must lie in line with d, there is of course no difficulty about drawing the second also. But as these are properties belonging to a very specially proportioned mechanism, and as in general we have no such helps to guide us, we shall suppose these properties to be unknown. We have, then, to start with, the positions of only the links OM and OM' in the second mechanism, with the lengths of all the other links. With known radii a_1, b_1, and c_1 respectively, we can draw the arcs of circles p, r, and s, and we know that on these arcs respectively must lie the points P, R, and S. We know, also, the distances between these points, and their relative positions (namely, here all on one straight line). The position of the link d_1 is uniquely determined by these conditions, but it cannot be found by any simple construction, or indeed any construction whatever that does not involve very complicated curves. Practically it is best found (except in such special cases as this, where certain characteristics of its motion are known from general reasoning) by a process of *fitting*. A template, here a straight edge (conveniently a strip of paper), is marked with the three points, P, R, and S, at the proper distances apart, and the strip is moved about until a position is found for it in which the three points lie simultaneously upon the three circles to which they correspond. The position of d_1 being thus found, the links a_1, b_1, and c_1 can at once be drawn, and the problem is solved.

It requires to be specially noticed, however, that the necessity of the "fitting" process does not essentially characterise this combination of links, but depends on the particular links whose position are given as data in the problem. Thus if d and d_1 were given instead of d and c the point O could be at once found, so that we should have

two links in each double kite (a and d in the one, a_1 and d_1 in the other), by the aid of which the whole mechanism could be drawn in the usual way. Similarly if d and a_1 were given, or d and c_1, or either of several other combinations, the whole mechanism could be drawn at once without "fitting." But in practical cases only the position of a fixed link and *one adjacent to it* can be given. Positions of other links can seldom be directly known. In any case the whole of the forty-five virtual centres can be drawn by the methods already given, and without difficulty of any kind. It may be said, therefore, that as a *kinematic chain* (p. 62), this combination of links does not differ essentially from the most simple class. But considering the *mechanisms* that can be obtained from this chain, and remembering that one of the given links in the instances mentioned above must always be the *fixed link* of the mechanism, we may say that these mechanisms belong to a higher order—we may call it a **second order** among mechanisms. The characteristic of this order is in every case that unless the positions of certain special links relatively to the fixed link form part of the data, the position of the mechanism as a whole cannot be found by ordinary line and circle constructions, but requires either the fitting process or the drawing of complex curves to be employed.

The ordinary "link motion" of a steam engine (Fig. 235), falls into precisely the same category as the last. It is a chain of six links, four of which (a, b, c, and d,) form a simple quadrilateral or lever-crank. One of these four (a) is paired at a point O, not upon its axis ($Q\ R$), with the fixed link f, and the opposite link c is connected to f at S by a single link e, attached to it at some point P, also in most cases not upon its axis (MN). Fig. 236 shows the whole combination put in a more schematic form, but in reality

without any alteration from the proportions of the last figure. All the virtual centres of this chain (fifteen in number) can be found in the usual way. The chain may be said to belong to the first class as a chain, but exactly as in the last case some of the mechanisms formed from it belong to the second order. Thus with the link f fixed, and varying positions of a given, we cannot find the positions of the other links directly, but must have recourse

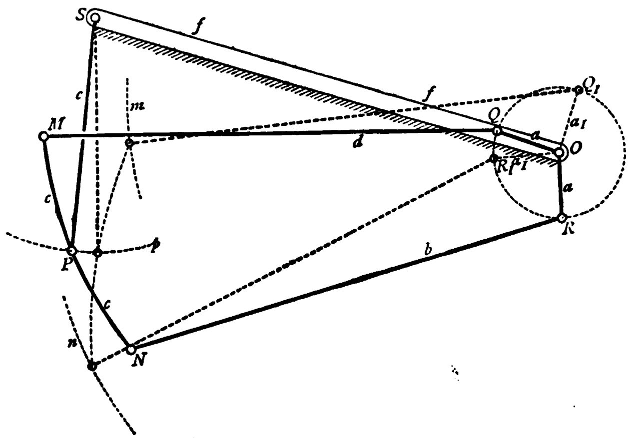


FIG. 235.

to the fitting process. Suppose, for instance, the position of a to change to a_1, so that we know the end points Q_1 and R_1 of d and b respectively. With these points as centres, and with radii equal to the length QM of the eccentric rods, we can draw the arcs of circles m and n. Also about S with radius $SP = e$, we can draw the arc p. We then know that the position of c must be such that its three points M, P, and N, must lie on the three arcs m, p, and n respectively, and

this position (which fixes the positions of *d*, *b*, and *e*) can be found in the way already described. It is shown dotted in the figure.

The block and pin by which the slide valve is driven in an engine form no part of the actual link motion—they are merely additions to it, which do not affect in any way the movements of its parts. In any such motion as that shown in the figure the valve is driven from a point which is guided along one straight line by a sliding pair. The nature of the connection is sketched in Fig. 237. *f* is the valve-rod guided by the pair 1; *g* a pin (called the "gab"-pin) in the valve-rod, which forms a pivot for a slider *h*, itself paired

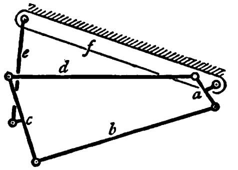


FIG. 236.

with the (curved or straight) link *c*. If a series of positions of the link be found, it will be seen that the position of *f*, and therefore of the slide valve, depends upon the position of the point in which the axes of the link and of *f* cut each other. It will further be seen that this is not always the same point *of the link*, although it is always the same point of the valve-rod, namely, the centre of the pin *g*. In other words, the swinging of the link, and chiefly the part of its motion due to the curve described by the end *P* of the suspension rod, causes it to move relatively to the block *h*. This motion is called the *slip* of the link; it is not only useless but very detrimental, and it is one characteristic of a well-designed link motion that it is reduced to the very

smallest possible dimensions. For this purpose, and to ensure that the valve-rod can receive from the eccentric as nearly as possible the motion which it would receive were the connection between them direct, the arrangement often takes the form sketched in Fig. 238, where the block *h* is replaced by cheeks embracing the link, and forming part of the pin *g*, which is expanded (see § 52) sufficiently to give the necessary metal for them. By using this construction the axes 2 and 3 can be made to coincide, which is impossible in Fig. 237.

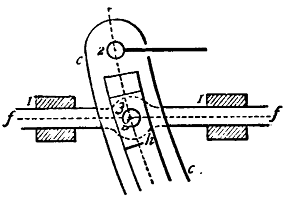


FIG. 237.

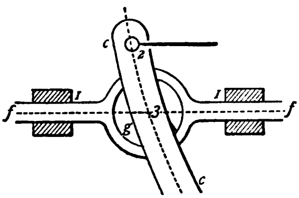


FIG. 238.

The principal statical problem connected with the link motion happens to be extremely easy of solution by the methods already given. It is this: given a force f_c (the valve resistance) acting on the link c in a given direction, to find the effort f_a in a given direction at the eccentric centre necessary to balance it. The problem is illustrated in Fig. 239. We have to deal with the three links a, c, and f respectively, the eccentrics (which, being rigidly connected together, form only one link), the "link" itself, and the frame of the engine. The virtual centre of the eccentrics relatively to the frame is of course at O_{af}, the centre of the shaft. The common point of a and c is at the join, O_{ac}, of the two eccentric rods. The virtual centre of the link relatively to the frame, O_{cf}, must lie upon the line O_{af} O_{ac}.

But it must also lie upon the line $O_{ef}\,O_{ce}$, which is the axis of the suspension link. Its position is therefore as marked on the figure. In all ordinary cases all three points are conveniently accessible, and as they can be marked by drawing only one line which is not already on the paper (namely, the line containing the three centres), the solving of the problem happens to be peculiarly easy in spite of the apparent complexity of the mechanism. The construction is shown complete in the figure, the lettering being the same as that formerly employed in Figs. 122 to 126, § 40.

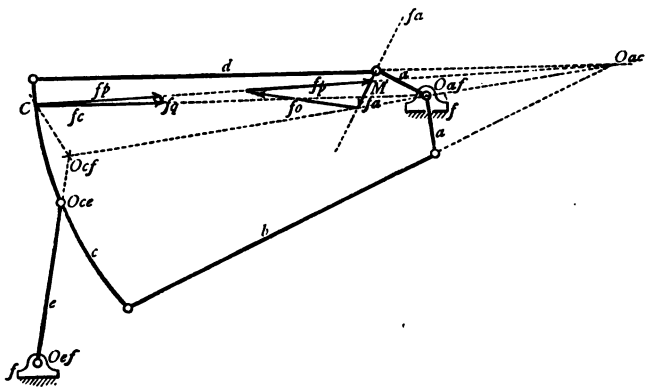


FIG. 239.

If either of the four links of the quadrilateral, *a*, *b*, *c*, or *d*, were fixed, and the position of any other one of them given, the position of all the other links of the mechanism could be found without fitting, which becomes necessary only when either *e* or *f* is the fixed link.

With the *Joy* valve-gear, shown in outline in Fig. 240, the condition of things is reversed. The chain as used by Mr. Joy has all its positions determinate without fitting,

but a number of its inversions (which, however, have not at present any practical value) require treating like the link motion. It may be interesting here to notice the general nature of this valve motion, which, when well proportioned,

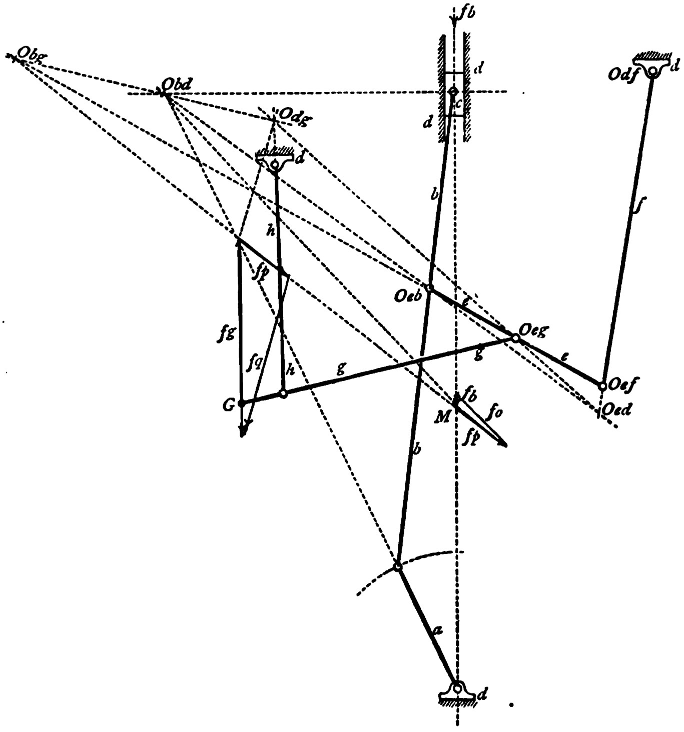


FIG. 240.

gives capital results. Its foundation is the slider-crank, *a*, *b*, *c*, *d*, the main driving chain of the engine. To this mechanism four links are added. Two of these, *f* and *h*,

are levers swinging about pins attached to the fixed link d; of the other two e is paired directly with f and b, and g with e and h.[1] The valve is driven in a direction parallel to that of the motion of c from some overhung point G upon the link g. An examination of the special characteristics of the motion of G, which suit it for working a slide valve, would involve too many technical points for our present purpose. But we may give the construction for solving the same statical problem as that considered in connection with the link motion in Fig. 239. Let f_g be the known valve resistance acting at G, it is required to find the necessary driving effort f_b, acting on the link b, the connecting rod of the engine, to balance it. We require for the solution the three centres O_{bd}, O_{dg}, and O_{bg}. The position of the first of these, O_{bd}, we have at once, although in many positions it will be inaccessible. The axis of the link e contains the points O_{eb} and O_{eg}, it must therefore be a line on which O_{bg} also lies, and by similar reasoning we know that O_{dg} must lie upon the axis of the link h. By drawing the line $O_{bd}\ O_{eb}$, we get the point O_{ed} on f, and by joining this to O_{eg} we find O_{dg} on h. The third point O_{bg} can then be found by drawing $O_{bd}\ O_{dg}$ to its join with the axis of e. Given these three points the construction is precisely as before (see § 40), and as the same letters are used in the figure it is not necessary to go over it again. In cases where the point O_{bd} is inaccessible, it will generally be most convenient to find the point O_{bg} separately in the same fashion as O_{dg} has been found above, and to arrange the construction so that the point M comes to O_{ab}, in the manner described on page 290, etc.

[1] For description of this gear, and some discussion on its proportioning, see Mr. Joy's paper in the *Proceedings of the Institution of Mechanical Engineers*, 1880, pp. 418-454.

Still limiting ourselves to mechanisms containing only lower pairs of kinematic elements we may sum up as follows. In mechanisms of what we may call the **first order**, if we are given the lengths of all the links, and the relative positions of any two of them, we can at once find the positions of all the others by direct line and circle constructions. In mechanisms of what we may call the **second order** (namely, those which we have been examining in this section), this is only possible if the relative positions of *certain* pairs, not of *any* pair, of links are given. Otherwise the positions of the remaining links can only be found by fitting—this process taking the place of the extremely complex geometrical construction which would otherwise be necessary. In both cases all the virtual centres of the links can be found by direct constructions involving only straight lines, so that we may say that as *kinematic chains* both orders of mechanisms belong to one class, which we may call the **first class.**

§ 59.—ORDER OF MECHANISMS. CHAINS CONTAINING HIGHER PAIRS (WHEEL-TEETH, CAMS, &c.).

THE great source of simplicity in mechanisms containing only lower pairs is that the virtual centres of adjacent links are always permanent centres (p. 71), or points occupying certain definite positions on those links. Directly we pass to mechanisms containing higher pairs, such as those of Figs. 241 and 242, we lose this simplification. Thus the points O_{ab} in Figs. 241 and 242, are not fixed points in a and b respectively, like the crank-pin centre in a slider-crank,

but vary their position in these links as the links vary their position relatively to each other. In both cases the virtual centres can be determined in the usual way, but the position of all the links can be directly determined without use of the fitting process only from given positions of a *certain* pair of them. Thus in Fig. 241, although the mechanism has only three links, the cam *a* and the link *b* paired with it must be the pair whose positions are given, if the position of the third is to be directly determined. Given any positions of *b* and *c*,

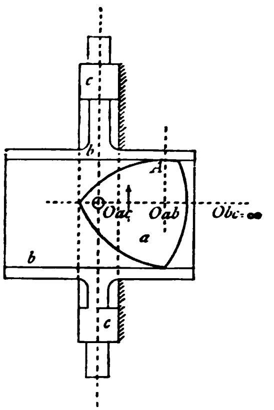


FIG. 241.

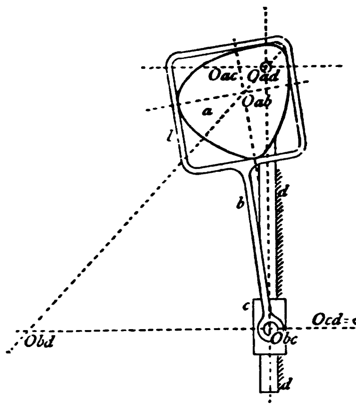


FIG. 242.

we know the position of the point O_{ac}, but the position of O_{ab} is not known, nor the particular point on *a* which becomes O_{ab} for the given position of *b*, and the position of the cam *a* requires to be found by a process of fitting essentially the same as, although different in detail from, that described in the last section. Or given any positions of *a* and *c*, in order to find that of *b* we require to "fit" the slot or groove in *b* over *a* in its given position. In practice this

would be done by merely drawing tangents to *a*, without the construction of a "template" as in the last section, but geometrically the process is the same. The tangent is drawn by eye only, and the position of *b* is therefore not found with any greater or different degree of accuracy than the position of the link in the link motion, Fig. 235.[1]

Exactly the same thing is true of the four-link mechanism of Fig. 242. In order to draw the positions of the links without fitting, we must have as data the relative positions of the two links connected by the higher pair, here *a* and *b*, the cam and the connecting rod. If the positions of any other pair of links, as *a* and *d*, be given, the positions of the remaining links can only be found by fitting *b* upon *a*.

In both cases all the virtual centres of the links are completely determinate at once.

These cam trains, therefore, are kinematic chains of the first class, while the mechanisms formed from them belong to the second order.

Ordinary toothed-wheel trains, looked at from our present stand-point, have some new interest and perhaps complexity. We have already seen that the virtual centres of such trains are very easily determinate; as kinematic chains, therefore, we may include them in the first class. Their classification as mechanisms is not so obvious. Fig. 243 represents the simplest form of wheel train, consisting of a frame *a*, and two spur wheels, *b* and *c*. Given the position of *a* and ther adii of *b* and *c*, it seems at first sight as if we could draw both those links at once. That this is a mistake will be seen as soon as one asks oneself for the actual position

[1] Of course in certain cases it may happen that the cam outline is a circle of known centre, or other curve to which a tangent can be accurately drawn with ease. But these are special cases.

of any particular points on the links, as *B* and *C*. If only *a* is given, the relative positions of these points are quite indeterminate, and therefore the relative positions of the links to which they belong. All that has been determined is the relative position of their centrodes, the pitch circles, and the symmetry of these prevents our going further in the way of connecting them with definite parts or points on the links whose motions they represent. But the want of determinateness goes even further than this. Given the relative positions of *a* and *b*, the latter as determined by the position of one point *B* in it besides its centre, the position of any definite point, as *C*, in the third link *c* still remains

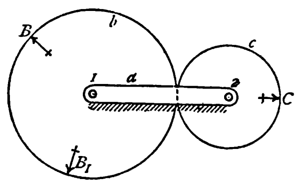


FIG. 243.

indeterminate. The third link may, in fact, occupy any angular position whatever relatively to *b*, so that the want of determinateness is real, and does not merely belong to our particular handling of the problem. But the mechanism is a completely constrained one, so that it must be possible, as it is clearly necessary, to state the problem in a different way—in some fashion, namely, which shall include the case of the former mechanisms without excluding this one. The difficulty is entirely met if we state our problem in the alternative fashion given at the beginning of § 58, which is equivalent to adding to our data the chain given complete in any one position. It was not necessary to refer to this

formerly, because no use was made of it; the mere knowledge of the lengths, etc., of the links carried with it the possibility of setting the mechanism out uniquely. Here, on the other hand, the mere knowledge of the dimensions of the links is not, as we have seen, sufficient of itself to determine uniquely any position of the mechanism, but only to enable us to find an infinite number of positions any one of which is possible.

If now we suppose given the mechanism of Fig. 243 with certain definite relative positions of the links b and c, as fixed, say, by the positions of the lines $1B$ and $2C$ upon them; and if then there is given any other position of b relatively to a, as $1B_1$, the angle $B1B_1$ being known—the position of c (*i.e.* of the line $2C$ in c) can be uniquely determined.[1] But, unless it chance that the ratio $\frac{2C}{1B}$ is some easily-handled whole number, the position of c requires for its determination either a fitting process, which would here involve the rolling on each other of templates of b and c, or, preferably, the substitution for fitting of some approximate construction for setting off on c a circumferential distance equal to BB_1.

If the relative positions of the two wheels be given as data, instead of the relative positions of the frame and one wheel, the position of the frame is obviously known at once without any fitting.

We thus find these mechanisms belong to the second

[1] It is necessary to make the condition as to $B1B_1$ being known, because the value of that angle may be not simply $B1B_1$, but $B1B_1 + n360°$, the wheel b having made any number of turns before being brought into the position $1B_1$. Thus there are a very large number of possible positions of $2C$ corresponding to $1B_1$, and the right one can only be known if the actual extent of angular motion of b between $1B$ and $1B_1$ be given.

order, exactly as the cam trains, the fitting process being necessary unless the relative positions of the two links connected by higher pairing, namely, here the two wheels, be the data.

In such a chain as the link motion of Fig. 235 some of the mechanisms belonged to the first, and some to the second, order. For if, *e.g.*, either of the links *a*, *b*, *c*, or *d* had been the fixed link the positions of all the others (including the links *e* and *f*), could have been directly determined if that of *any one* of them had been given. In the cases now before us, however, *all* the mechanisms belong to the second order, for, whichever link be fixed, direct determination of the positions of the others is only possible if the data include the position of some particular one of them.

If we consider compound wheel trains, as those of Figs. 65 and 66, § 19, in the same way, we find they fall into the same class and order. For we know that in such trains the relative motions of any pair of wheels, by however many intermediate wheels they may be separated, are fully represented by those of *one pair* of pitch circles of determinate diameters working in direct contact. As regards any one pair of its wheels and the frame, a compound wheel train therefore reduces at once to the simple train just examined. If the relative positions of *all* the wheels be required, the fitting process, or its equivalent already mentioned, has to be resorted to for each pair separately. The sun-and-planet motion (Fig. 73), and other familiar combinations of linkwork and wheel gearing, belong to the same class and order with those just considered.

Higher chains and mechanisms occur so seldom in practice [1] that it will be sufficient to give one example, that shown in

[1] Limiting our statements here, as always, to mechanisms having plane motion only.

Fig. 244. This is a mechanism which can scarcely be said to be used in machine construction, but which is sometimes met with in collections as an illustration of certain special motions. It consists of a fixed link or frame, *g*, containing a straight slot and two pins. About the two latter turn wheels, *a* and *f*, equal or unequal in diameter. Crank pins in these are connected by rods, *b* and *e*, to the two ends of a beam, *c*, which

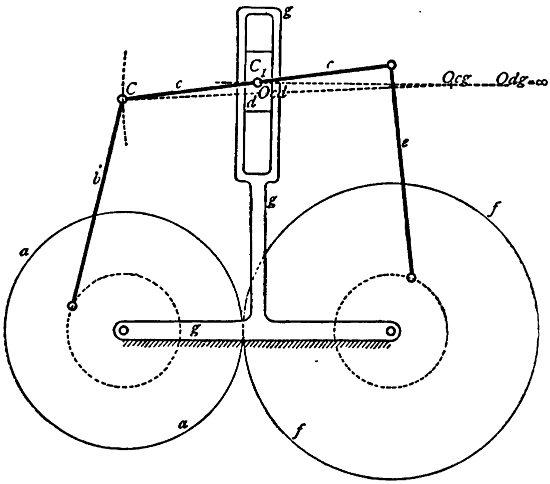


FIG. 244.

is connected at its centre, or any other convenient point, to a block, *d*, sliding in the slot already mentioned in the frame *g*. If the two wheels were equal, the two crank pins placed symmetrically to the vertical axis, and the beam *c* pivotted at its mid-point, the beam would simply move up and down, remaining always horizontal. The pin connection between *c* and *d* would become superfluous, as would also the spur

wheels, and the mechanism would become simply a doubled slider crank, as sketched in Fig. 245. All points in the link *c* would move in parallel straight paths, the link as a whole reciprocating through a constant stroke exactly as in a steam engine. In the more general form of Fig. 244, however, the motion of *c* is much more complex, its different points having all different motions, that of the point *c* being the one usually studied. This point is constrained by the sliding pair to move always up and down along one straight line, but the distance which it moves along that line varies

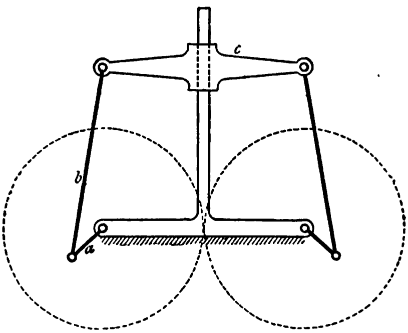


FIG. 245.

at each reciprocation, its "stroke" changing gradually from minimum to maximum, and *vice versâ*, as the phases, or relative positions, of the cranks alter. If a pencil were attached to *c*, and caused to describe a curve upon a sheet of paper moved uniformly, say by gearing, from either of the wheels from right to left, the curve traced would be of the kind shown in Fig. 246, where the varying vertical height shows the varying stroke of the reciprocating point C_1.

In this mechanism it will be found that we have to do

with entirely different conditions from those hitherto examined. The relative positions of *no* two links whatever enable us to find the positions of all the rest (the lengths of all being, of course, supposed given as before), *without* fitting, and although the relative positions of any pair of links enable us to find possible positions of the rest by using that process, yet with only certain pairs do we know that such positions are consistent with any given starting positions of the mechanism. In the case before us, for example, the whole mechanism can be drawn most readily from the given positions of two of its links, if these two are the two spur wheel cranks *a* and *f*. The frame can then be drawn at once, and the position of the beam *c* found by fitting. In this case we do not even require to make use of the given

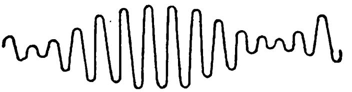


FIG. 246.

position of the whole mechanism which we have assumed to belong (p. 439) to our data. By making use of this starting position, and the known angle between it and a given position of either of the wheels, we can always find the position of one wheel from that of the other, if the frame also is given. Hence a given position of either wheel along with the frame is sufficient for the determination of the positions of all the other links, the fitting process being used twice. If the data be the positions of one of the connecting rods and the frame (an improbable combination) one of the cranks can be drawn directly, and therefore both the wheels and then the rest of the mechanism found as before. If the data be the positions of the beam and the frame,

possible, but not unique, positions of the other links can be found. For the connecting rod lengths, swept from their given upper points as centres, will cut each crank circle twice. We get, therefore, four possible positions of the mechanism as a whole, of which in the general case no one is of necessity compatible with any given starting position of the whole chain. If the data were the positions of the two connecting rods *b* and *e*, it will be found that fitting gives in general two pairs of positions for the cranks, neither of which is necessarily compatible with a given starting position of the mechanism. If, lastly, the data be the positions of the slider *d* and the beam, an indefinite number of different possible positions of the mechanism can be found by fitting, and without further data it is not possible to say which out of these positions is consistent with a given starting position. This chain, then, differs from the former in that whichever of its links be fixed, the positions of the remaining links (that of one being given) can only be found by fitting, and also in that unless certain particular links be used as the data, the position so found may not be compatible with any given starting position. We may consider such mechanisms to be of the **third order**.

Of the twenty-one virtual centres in this chain, nine are between adjacent links, and these nine only can be found by our former construction. One line can be found on which each of the remaining twelve centres lie, but the position of no one of them can be directly obtained. We may consider this to separate the chain from all former ones sufficiently to make it a type of a **second class** of kinematic chains. To find any one of the undetermined virtual centres it is necessary to find the path of one point in the link to which it corresponds, and then to construct (of course approximately only) the normal to this path. Thus in the figure

the path of C (relatively to g), has been determined, and a normal drawn to this must be a line including O_{cg}. But this point must be on a line including O_{cd} and O_{dg}, both of which we know; its position is therefore completely determined. Any one of the twelve virtual centres having been found in this way, all the others can be found at once by the usual constructions.

§ 60.—RATCHET AND CLICK TRAINS.

In speaking formerly of the condition of constrainment in a mechanism, we qualified a statement (see p. 59) by saying that it referred to "those links in which motion was possible at any instant." This qualification was made in view of the fact that there are a number of mechanisms in which special provision is made for stopping the motion of one or more links entirely (or in one sense only) at regular or irregular intervals. Most of the contrivances for carrying out this object fall under the head of **click** or **ratchet gear.** Reuleaux has classified these gears somewhat elaborately in his *Kinematics of Machinery*,[1] and it is not necessary to go over the same ground here, as they do not present any mechanical problems essentially differing from those already examined. In cases where a click is arranged to prevent the motion of a body entirely (as b with a, Fig. 247), it simply makes that body, and perhaps others also, part of the same link with itself. In the case where it prevents motion of the body in one sense only (as b prevents the downward, but not the upward, motion of a in Fig. 248), it becomes one with that body only so far as those forces tending to move it in one sense (here downward) are concerned. If the link a in

[1] §§ 119 to 121.

Fig. 248, for instance, be lifted upward by any force, *b* loses at once its fixing action, and moves about its fulcrum in *c* in just such fashion as corresponds to the higher pairing between the teeth of the rack and the end of the click. Exactly the same thing is true in the case of Fig. 247, where a ratchet wheel takes the place of the rack.

FIG. 247.

FIG. 248.

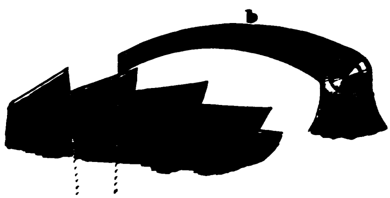


FIG. 249.

In the common form of ratchet train shown in Fig. 250, the click *b* acts just as in the last case, alternately forming *a*, *b*, and *c*, into one link, and being caused (by the rise of *a*) to swing about its fulcrum on *c*, remaining in contact with the teeth of *a* either by its own weight or by some other form of "force-closure" (p. 393). The click or ratchet b_1, in exactly corresponding fashion, alternately makes *a*, b_1, and c_1 into one link, and swings about its fulcrum on c_1, rubbing against

the teeth of a just as the other. Thus when c_1 is lifted, the rack a is lifted also, through b_1, as if it were part of c_1, the retaining click b being idle. But when c_1 is lowered, the rack a does not fall also, because the click b comes into action, and temporarily fixes it to the frame c. The rack,

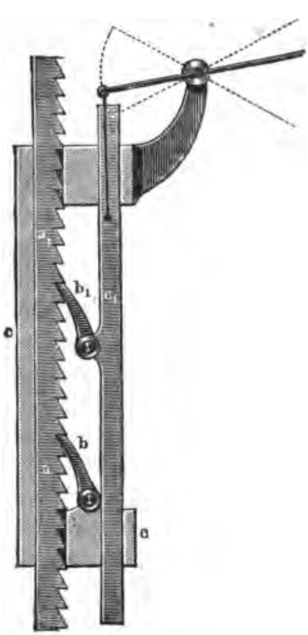


FIG. 250.

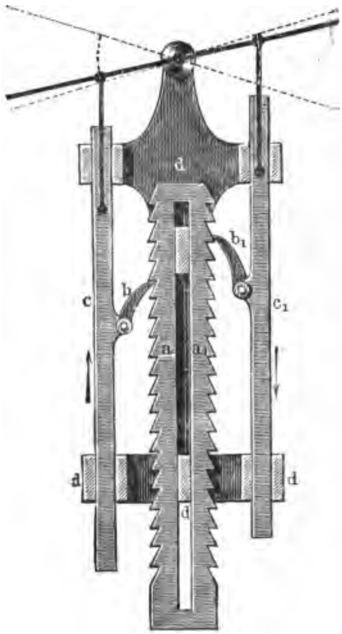


FIG. 251.

therefore, receives an ***intermittent*** upward motion, while the driving link c_1 has a ***continuous*** reciprocating motion. By the use of a double ratchet train, as Fig. 251, the rack a may be made to receive an upward motion for both swings of the driving arm.

Perhaps the only point about these mechanisms calling for further special mention here is that it is essential to their proper, and indeed safe, working, that the form of the faces of the ratchet teeth be so designed that under no circumstances can direct pressure between tooth and click or ratchet tend to throw the latter out of gear. This matter will be further considered under the head of Friction, in § 71.

§ 61.—CHAINS CONTAINING NON-RIGID LINKS.

In our original definition of a machine (p. 2) we spoke of the bodies of which it consisted—its "links"— as **resistant** rather than **rigid**, because there are several important cases in which non-rigid bodies can be and are used in mechanisms or machines, without destroying the required constrainment of their motions. The non-rigid, or simply resistant, bodies used in practice may be classified as (i) fluids, (ii) springs, and (iii) belts, cords, chains, &c. In part they affect our problems very little indeed; in part they introduce new and complex problems belonging in reality to the subject of elasticity, and, therefore, beyond our present work. We shall here only briefly notice the general points connected with the use of each class of resistant bodies.

(i) **Fluids.**—*Water* is occasionally used as a part of a machine to transmit pressure to long distances and through any change of direction. This it can do with considerable exactness in virtue of its (comparative) incompressibility, if it is free from air. But the invariable presence of small quantities of air in the water itself, and the difficulty of ensuring that no additional air shall be pumped into the pipes along with the water, makes any "water rod" arrange-

ment suitable only for cases where the transmission of a quantity of *work* is the chief matter, and no exact transmission of *motion* is required. Occasionally glycerine is mixed with the water to prevent its freezing, and sometimes for the same and other reasons oil takes the place of the water. In a very few cases a column of *air* is used instead of the water, but in this case the transmission either of work or motion becomes exceedingly inexact on account of the compressibility of the air, and the changes of volume which it undergoes with changing temperatures.

(ii) **Springs.**—In the largest sense of the word every link in a machine, being made of elastic material, is a spring. Each is stretched, compressed, twisted, bent, or in some fashion strained, by every load that acts upon it, and the strain,[1] whatever it may be, is very closely proportional to the stress in the material. Thus the alternate extensions and compressions of a piston or connecting rod, if they could be conveniently measured and recorded, would enable us to find the work done in a steam-engine cylinder just as correctly and completely as an indicator card. Or, as Hirn has shown,[2] the beam or the shaft of an engine may be used as the spring of a dynamometer, to measure—by its recorded deflections or twists—the work being done by the engine. But in all such cases the strain is exceedingly small in proportion to any of the motions of the different links of the machine, and has to be made visible and measurable by some special arrangement of exaggerating apparatus. The name *spring* is not generally applied in such a case, but is restricted to those bodies or links whose strain under load is

[1] By *strain* is meant always *alteration of form*, not force causing alteration of form, nor molecular resistance to alteration of form, for which latter we have kept the word stress (p. 261).

[2] See, for example, his *Les Pandynamomètres*, Paris, 1867.

not only proportional to the straining force, but is comparable in extent or dimension to the proper motions of the links of the machine. Springs in this sense of the word are used generally for one of three purposes, either (*a*) to measure force, (*b*) to limit force, or (*c*) to store up work or energy. Class *a* is represented by all steam-engine indicators and many dynamometers—mechanisms in which the motion of one link, viz., the spring and any parts in rigid connection with it, while constrained in *direction* in the same way as the motion of every other link, is made to be proportional in *magnitude* to the force acting on itself. In the instruments named the spring is made to show, or to record (or both), the extent of its motion. If the record be made, as it usually is, upon a sheet of paper caused to move at right angles to the direction of motion of the spring, and at a rate proportional to the rate of motion of the body whose resistance is measured by the spring, we obtain for record such a curve as is shown in Fig. 252. The ordinates of the curve, as $A\,A_1$ or $B\,B_1$, are proportional to the motion of the spring, and, therefore, represent pressures or forces. The abscissæ, as $O\,A$ or $O\,B$, are proportional to the distances moved through by the body on which the forces are acting. The *area* under the curve, as $A\,A_1\,B_1\,B$, represents therefore, measured on suitable scales, the product of pressure and distance, or *work*. Thus the area $A\,A_1\,B_1\,B$ represents the amount of work done on or by the moving body in passing through the distance represented by $A\,B$. In the case of a steam engine we have not a body moving unlimitedly on in one direction, but one which has a short stroke only, returning always to its original position. In such a case the diagram of work traced by the means just described takes some such form as is shown in Fig. 253. The work done by the steam on the piston is shown by the area

F B C D E. The work done in returning by *the piston on the steam* is shown by *D A F E*. The net work done is the difference between these two areas, or *A B C D*, which is all that the steam-engine "indicator" shows in drawing the "indicator card" *A B C D*, which we have already made use of in § 47.

In steam-engine indicators the spring is invariably of the type known as "spiral"—a coil of tempered steel wire twisted helically, and compressed or extended in the direction of the axis of the helix. Fig. 254 shows one of the most recent, and probably the best,[1] form taken by the spring of an indicator, the peculiarity about it being

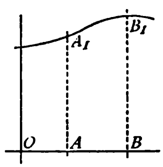


FIG. 252.

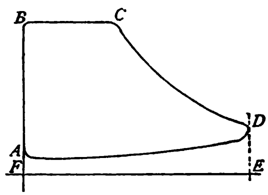


FIG. 253.

the symmetrical arrangement of double coil or helix, with a central button at the lower end to ensure true axial direction of pressure. In dynamometers flat or bent plate springs are often used instead of spiral springs.[2] This is the case, for instance, in most of the dynamometers of the Royal Agricultural Society, on which so many valuable experiments have been made.

It is necessary to bear in mind that, although in all these

[1] This is the form of spring used in the "Crosby" indicator. See, for example, *Engineering*, vol. 37, p. 185.

[2] See, for example, Mr. W. E. Rich's paper in the *Proceedings of the Institute of Mechanical Engineers*, 1876, pp. 199-227.

cases the motion of the recording point attached to the spring is as simple to deal with as the motion of any other point in a mechanism, yet the motions relatively to each other of different points in the spring are extremely complex. For here we are dealing with a body in which the deformation or strain is no longer intentionally kept as minute as possible (p. 261), but intentionally made as large as possible. And in order that we may know beforehand how much the recording point of a given spring will move

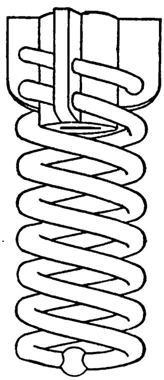


FIG. 254.

under a given load, it is necessary that we should first know the laws according to which the deformation of the spring, as a whole, will take place when it is subjected to pressure. These laws are usually complex, their consideration belongs to the theory of elasticity, and goes beyond our present limits. It may be said about them, however, that although they have been pretty fully worked out from the mathematical side, reasoning upwards from certain definite and comparatively simple assumptions, there yet remains much

work to be done in connection with them before they can be taken to represent the actually occurring and very complicated physical phenomena.

Of springs used to *limit* force the safety valve of a locomotive affords a very familiar example. The spring in this case is a spiral spring so compressed as to act on the valve, when resting against its seat, with a certain definite pressure. So soon as the pressure of steam below the valve exceeds the pressure of the spring above it, the valve moves upwards, and by so doing allows steam to escape,

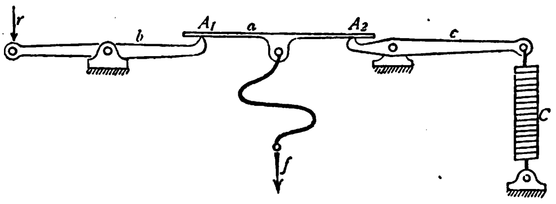


FIG. 255.

and the pressure to fall,[1] or at any rate prevents the pressure from rising. But a spring safety valve can hardly be included among mechanisms or machines, although it is essentially a machine while it is opening or closing. A more interesting example of the use of a spring to limit a pressure in a machine proper is shown in Fig. 255, which represents a portion of one of the testing machines used in Wöhler's experiments on the "fatigue" of metals.[2] Here it

[1] This must not be taken as a description of the behaviour of any *actual* spring safety valve, but only as a simplified statement of its ideal behaviour.

[2] See *Ueber die Festigkeitsversuche mit Eisen und Stahl*, A. Wöhler, Berlin, 1870, and also *Engineering*, vol. xi. pp. 199, etc. ("Fatigue of Metals").

is desired that a certain force, f, liable to irregular variations, should exert at r an effort not exceeding a definite invariable magnitude. For this purpose the force f is caused to act on the lever b (and therefore at r) through the intervention of a crosshead a, one end of which, A_1, rests upon b, while the other end, A_2, rests upon a lever c. This lever is held in position by a spring, C, of definite resistance. So long, therefore, as the pressure at A_2 due to f is less than that due to the spring, the point A_2 cannot move. Any motion of a that occurs must be about A_2 as a fixed point, and the pressure at A_1 will be entirely transmitted to r, the resistances at A_1 and A_2 being equal. But directly the force f attains such a magnitude that its downward pressure on A_2 exceeds the fixed upward pressure there due to the spring, the point A_2, that is, the inner end of the lever c, drops, and the force f remains without increase. This absence of increase in f comes about by its transmission to a through a spring, the pull in which is released as soon as A_2 sinks. It is obvious that the action of this apparatus would not be correct if the motion of A_2 were at all considerable, because as it falls it extends the spring C, and therefore its resistance to falling increases. In any case the real maximum value of the pressure at A_2 (and therefore at A_1) is not exactly that due to C in its normal position, but in its most extended position. By judicious management it can be arranged that this quantity exceeds the normal pull due to C by a quantity not only very small, but also experimentally determinable with very considerable accuracy.

The *storing up of work* has been mentioned as a third use of springs—of which the buffer springs of a railway wagon afford perhaps the most familiar example. If a rigid body has to take up work in itself while undergoing mere elastic deformation, the inevitable smallness of such

deformation may cause the stress to rise to some most inconveniently high amount while still the work taken up is excessively small. In the case of a railway wagon, for instance, exposed to frequent and sudden blows from bodies moving with considerable velocity, the effect of such blows upon the frame of the wagon—if they were received directly by it—might and probably would be most injurious to it. It is therefore most common to provide spring buffers for the purpose of receiving such blows, storing up readily, and without any injury to themselves, the energy received by them from the striking body. For this purpose some body is required which can be made to change its form very greatly without loss of elasticity, and at the same time offer a sufficiently large resistance to the change. These requirements are exactly fulfilled by a stiff spiral steel buffer spring. Such a spring, five inches in diameter outside, and twelve inches long, may require a pressure of about four tons before it is compressed "home." The *mean* pressure during the compression is therefore two tons, and the amount of work stored up in the spring when compressed (it being then 6.5 inches long) is about eleven inch-tons. When it is remembered that a loaded railway truck weighing sixteen tons, and moving with a velocity of ten miles per hour, only requires to get rid of about 100 inch-tons of energy in order to bring it completely to rest, it will be seen how important a part a couple of such springs may play in absorbing energy which would otherwise tend to the rapid destruction of the wagon. The energy thus stored up in the spring has not, of course, vanished. The spring must, sooner or later, come back to its original condition of equilibrium. The work done on it and stored up in it may give negative acceleration to the striking body, or positive acceleration to the body to which

the spring is attached, or both, but in any case this can come about at (what we may call) the leisure of the material, by the action of a simple known force (the resistance of the spring to compression). In the absence of the spring it may often be impossible for a large mass, moving with a high velocity, to impart to the mass of another large but stationary body, with which it suddenly comes into contact, a sufficiently great acceleration to keep the mutual pressure of the two bodies one on the other within such limits as will ensure that neither is fractured nor seriously injured.

In this case, therefore, springs are used to store up work which might otherwise cause injurious stresses, the work being promptly re-stored, and expended in causing harmless acceleration. An exactly similar case occurs in connection with the "spring beams" and buffer blocks of a Cornish engine (p. 336). But in most machines where springs are used in this fashion, the energy stored in them is not derived from the momentum of some rapidly-moving body, but directly from work done on some other part of the machine. And the acceleration of some part or parts of the machine due to the re-storing of this energy is often not the result directly wanted, which is commonly no more than a rapid change of position of those parts. The change of position, that is, is the thing essential to the machine at the particular instant—the rate at which that change is effected may be immaterial. The springs used in connection with the valves of Corliss engines, of which one arrangement is sketched in Fig. 256, form an illustration of this. In this arrangement *a* is a lever receiving a continual reciprocating motion from the engine through the rod *h*. On its back is a flat spring, *f*, which has, by the action of the gearing, been drawn close up to *a*, and in which, therefore, work has been stored up. The upper end of the spring is connected

by a link *e* to the rod *d*, which works the valve, and the lever *a* is at the same time connected to the same valve rod by the horn piece *c*. During motion in the direction of the arrow, *a*, *f*, and *c* all move as one link, pushing the valve rod *d* to the left, and opening the valve. When the time comes that the valve should close, some form of stop g_1[1] comes in contact with the upper part of the horn piece, stops its forward motion with *a*, and causes it to tilt over on its pin. This throws it out of gear with *d*, which is left free,

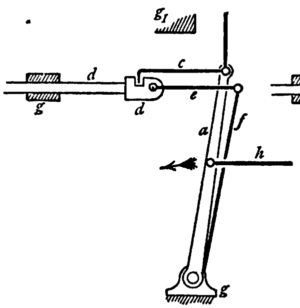


FIG. 256.

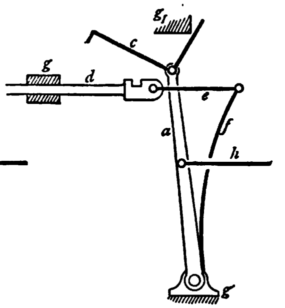


FIG. 257.

to be acted upon by the spring *f*. This spring has been pulling *d* backwards all the time, but its pull has been resisted by the catch *c*. This pull is now suddenly left unbalanced, and the spring, therefore, flies back into its unstrained position (as in Fig. 257), and pulls the valve rod *d* with it, in this fashion suddenly closing the valve by help

[1] This stop is in reality controlled in position by the governor, and is therefore movable relatively to the frame of the machine, but so long as the engine is working under constant resistance it remains steady, and we may therefore take it as being at any given instant a part of the fixed link.

of the work already stored up in it while it was being compressed.

(iii) **Belts, Straps, &c.**, form the third and perhaps the most important class of merely resistant bodies used in machines. Their use differs from that of springs in that their alterations of form under load are not directly utilised, but are, on the contrary, made to come in in such a way as to be fairly negligible in considering the motions of the mechanisms of which they form a part. Fig. 258 represents the ordinary strap connection between two pulleys. Kinematically it is intended that *b* and *c* should revolve in the

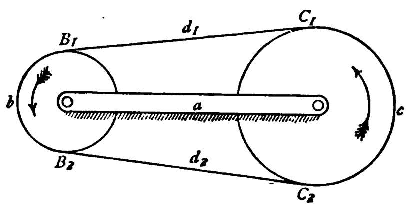


FIG. 258.

same sense, and with angular velocities inversely proportional to their diameters, just as if they were spur wheels connected by an idle wheel (p. 132). The wheels would actually do this only if the strap never slipped on either pulley, and if the length of strap between the points B_1 and C_1 remained always constant, as also the length of strap between B_2 and C_2. In ordinary good working, slipping is no doubt practically absent, and so far as this is concerned the required velocity ratio is accurately transmitted.[1] But

[1] See, for instance, the recorded revolutions during many hours' trial of an engine and the machines driven by it in experiments made by Messrs. Bryan Donkin & Co., at the South Metropolitan Gas Works, described in *Engineering*, vol. xxv. p. 117.

there may be continual small changes in the tensions in the two halves of the strap d_1 and d_2, which will have the same effect on the velocity ratio transmitted as in spur gearing would be due to the use of wrongly-shaped tooth profiles (p. 121). So long as the total length of the strap is not permanently altered, the effect of small changes of length will be to make small changes in the velocity ratio alternately above and below its mean value, but without change in that mean value itself. For practical purposes we may neglect these changes in belt gearing as we do in spur gearing.

In respect to transmission of *work*, however, belt gearing differs essentially from spur gearing in the amount of work which it absorbs itself. This is apart altogether from the question of the work taken up by friction in the bearings, which is more or less common to the two cases, although the existence of strap tensions, not directly dependent on the driving effort (or difference between the two tensions), makes of course an important difference (§ 78). The continual bending of the stiff strap round the cylindrical pulleys absorbs in itself in some cases a very considerable amount of work, and this work is, of course, not transmitted from the driving to the driven pulley. This waste of work is a condition, in this case, of the use of a non-rigid material, and has no counterpart in the rigid connection of spur gearing.

In pulley tackle, such as is shown in Figs. 259 and 260, we are again on the border line of apparatus which can legitimately be called a machine. In these cases the constrainment of motion is generally most imperfect. The tackle is required to lift some object of considerable weight, and to lift it with reasonable steadiness, but very frequently indeed it exerts a large sideway pull as well as a lift. Once the object is being fairly lifted, moreover, its swinging to and

fro is not considered to destroy in any way the action of the tackle. Pulley tackle, therefore (although in some form it is often included among the "simple machines" of § 51) cannot be considered sufficiently constrained in its motions to be suitable for kinematic examination. From a static point of view—neglecting friction and work done in bending the rope—it is simple enough. The weight W, in Fig. 260, depends from five equal cords—the fact that all five plies or "parts" are in reality portions of one and the same

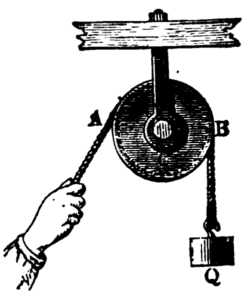


FIG. 259.

FIG. 260.

piece of rope leaves this unaffected. The whole weight is assumed to be equally distributed between the five plies, so that the tension in each of them is supposed to be equal, and the tension in the "fall" h is of course equal to the tension in the rest of the rope or $\frac{W}{5}$. Apart from friction then, a pull $\frac{W}{5}$ in the fall is sufficient to balance a weight W hanging from the lower or "running" block. From the

principle of equality of work, therefore, the fall must be pulled downwards five times as fast as the weight rises. In an actual, and not an ideal, pulley tackle, a very considerable effort must be expended in overcoming friction (see § 80), and not a little work has to be done in bending the rope round the sheaves. So that the pull in the fall in hoisting must be actually very much greater than the fraction of the load indicated by the numbers of plies supporting the load.

The motions in the pulley tackle are not really plane motions (p. 12), even looked at in the most ideal fashion. But as the actual non-plane motions of the different parts of the cord do not come into consideration at all, it has not seemed out of place to mention the tackle while speaking of belt gearing generally.

The theorem of the virtual centre applies only to rigid bodies; its existence presupposes (p. 261) that the particles of the body do not alter their positions relatively to each other. Therefore a non-rigid link in a machine has no virtual centre; different parts of it are moving at the same instant about different points. Force and velocity problems, therefore, so far as they concern non-rigid links, have to be worked out by considerations quite different from those hitherto employed. Some of these considerations have been mentioned above, others will be discussed further on.

Many of the most interesting and important problems connected with the equilibrium of forces and the transmission of work in belt gearing and pulley tackle, are so closely connected with *friction* that they must be postponed until that subject has been looked at in the next chapter. Some discussion of them will be found in §§ 78 to 80.

The so-called "flexible shafts," which are now found most useful in machine shops for driving small tools (drills

or borers for example) in more or less inaccessible situations, are further illustrations of non-rigid elements. As with spiral springs, the motions of different points in the shaft relatively to each other are excessively complex, although the motion transmitted by the shaft as a whole is only a simple rotation.

CHAPTER XI.

NON-PLANE MOTION.

§ 62.—THE SCREW.

IN the preceding chapters we have limited ourselves almost entirely to the consideration of mechanisms in which only plane motions occur. These form by far the largest and most important class with which the engineer has practically to deal. We have now to notice some of the principal *non-plane* motions utilised in machinery, and shall in the first instance examine those conditioned by the use of the screw and nut, Fig. 261.[1]

In § 2 we have already noticed the characteristics of screw motion, or twist; and in § 10 we have seen that this motion could be completely constrained by the ordinary screw and nut, a pair of elements which we classed among the *lower* pairs because of its surface contact. Familiar and important as this pair is, there is hardly an instance in which it is used for the sake of its own characteristic helical motion. With scarcely an exception the screw motion is resolved into its two components, rotation and

[1] A more general investigation of screw motion in mechanisms, of which this is the simplest (and a very special) case, will be found in §§ 68 to 70.

translation, and these two motions are employed separately on different links of the chain containing the screw. Fig. 262 shows the most familiar illustration of this. The screw forms

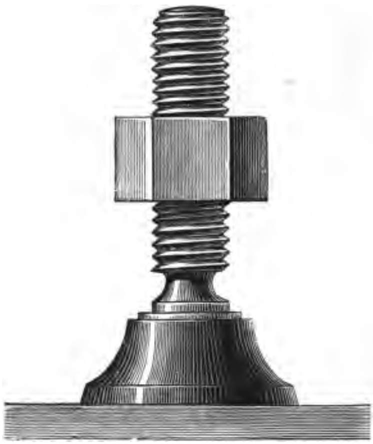


FIG. 261.

part of the link *a* of a three-link chain. The link carries also a turning element or pin, which is paired with *c*, and *c*, in turn, forms a sliding pair with the outside of the nut *b*.

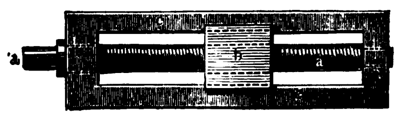


FIG. 262.

The motion of *a* relatively to *c* is a rotation,[1] that of *b* re-

[1] It is presupposed that suitable collars prevent any endlong motion of *a* in *c*.

latively to c a translation. The motion of a relatively to b is twist, but this is the one of the three motions of which no use is made under ordinary circumstances.

If the mean radius, or pitch radius, of the screw be called r, and the pitch p, then any point of the screw at a radius r will move through a distance $2\pi r$ relatively to the link c, while b only moves through a distance p relatively to the same link. Any such point will move $\frac{2\pi r}{p}$ times as fast (relatively to c) as b, and any force applied at it, in its direction of motion, will balance a resistance $\frac{2\pi r}{p}$ times greater than itself to the motion of b along c. As such a force can be readily caused to act at some radius R very greatly larger than r, without any alteration in the value of p, the screw presents the possibility of attaining in a small compass a very large "mechanical advantage," $\frac{2\pi R}{p}$. We shall see in the next chapter how very seriously this apparent advantage is reduced by unavoidable frictional resistances.

Fig. 263 shows a **screw press**, which is in reality exactly the same chain as the last figure with the link b fixed. The relative motions of the links remain exactly as before, but the twist of the link a becomes more obvious, as it occurs relatively to the fixed link. In such a case the driving effort upon a cannot remain in the same plane (as in the last case), but must change its position axially as the screw goes bodily down or up. Unless, however, the screw be moved by hand, in which case such a change of position does not require to be considered, means are taken to keep the driving effort always in one plane, so that again the actual screw motion does not come into practical consideration. Thus the arrangement of Fig. 264 is often used, in

which *c* is the fixed link, but *b* carries the screw instead of the nut, and *a* the nut instead of the screw. The screwed part of *b* merely slides in *c*, and the link *a*, which is prevented by shoulders from having an endlong motion, takes externally the form of a belt-pulley or a spur-wheel, which can be driven in the usual way. It is hardly necessary to point out that the interchange of the forms of screw and nut—external and internal screws—make no more change in the mechanism than the interchange of eye and pin in a turning pair, or slot and block in a sliding pair.

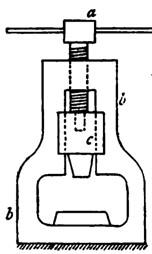


FIG. 263.

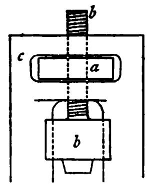


FIG. 264.

In the mechanisms formerly considered we had only to deal with forces acting in, or parallel to, one plane. All other forces or force components were, by hypothesis, balanced as they appeared by stresses in the links (p. 7). Here, however, it happens almost invariably that effort and resistance act in different planes. The effort in most cases (as in the last figure for instance) acts in a plane normal to the axis of the screw, while the resistance acts in a plane passing through that axis, very often, indeed, acting directly in its line. In any case we have still, exactly as before, the condition that all force components tending to cause motions which are incompatible with those permitted by the connec-

tion between the links, are entirely balanced by stresses in those links.

In the plane mechanisms hitherto studied we have assumed tacitly that the smallest force acting on any link, and acting in such a direction as to move that link, would move the whole mechanism. Apart from friction, this is strictly true, and under the same conditions it is true with screw mechanisms also. But here, as we shall see later on, the effect of friction is much more serious than where there are only pin-joints or even ordinary slides. With a screw of ordinary proportions, and working with ordinary lubrication, no effort, however great, acting on the nut in the direction

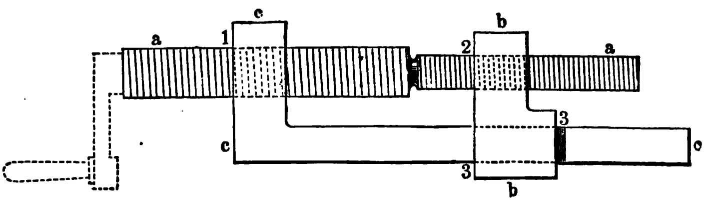


FIG. 265.

of the axis of the screw, could cause rotation of the screw. We therefore obtain here a condition, sometimes of great practical convenience, differing essentially from any with which we have had hitherto to do, namely, that as regards effort and resistance the mechanism is non-reversible. But as this condition depends entirely on frictional resistances its further examination must be deferred.[1]

We shall now notice briefly a few of the principal mechanisms in which screws are used.

The three-link chain shown in Fig. 265 finds several applications; it is commonly known as a **differential**

[1] See § 74.

screw chain. The link a consists of two screw elements, of different pitches; b carries the nut for one of these elements, and c for the other, and b and c are themselves connected by a slide. Let us write p_c for the pitch of a in c, and p_b for its pitch in b. Then if c be fixed, as is usually the case, each complete turn of a will cause it to move through a distance p_c relatively to c, and (simultaneously) through a distance p_b relatively to b. The corresponding motion of b relatively to c will be the difference or the sum of p_c and p_b, according to whether the two screws are of the same or of opposite hands. In the figure they are of the same hand, and the motion of b relatively to c for one turn of

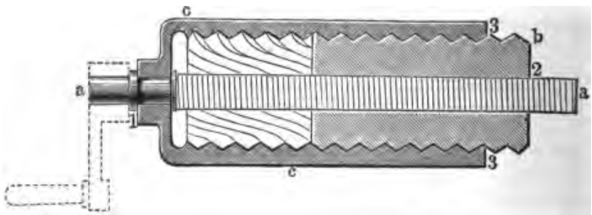


FIG. 266.

the screw is equal to $p_c - p_b$. The mechanism is therefore equivalent to that of Fig. 262, with a screw whose pitch was $p_c - p_b$. But this quantity may be excessively small, much smaller than it would be practicable to make the pitch of an ordinary screw, and in this way the "differential" arrangement may enable us to get a mechanical advantage much greater than could conveniently be otherwise obtained by a screw, the advantage being, of course, $\dfrac{2\pi r}{p_c - p_b}$.

In the chain of Fig. 266 the link b carries two screw elements of different pitches, with which a and c are paired. The con-

nection between a and c is a turning pair. If here c be fixed, the motion of b relatively to it must be twist, but its rotation must be as much slower than that of a as p_a is less than p_b. Thus in one turn of a the link b receives an axial motion of p_a, while at the same time its whole motion (relatively to c) is a twist whose pitch is p_b. It must therefore have made only the fraction $\frac{p_a}{p_b}$ of a complete turn. In a similar way the relative motions of the other links can be examined, their static relations following most easily, as before, from their relative velocities.

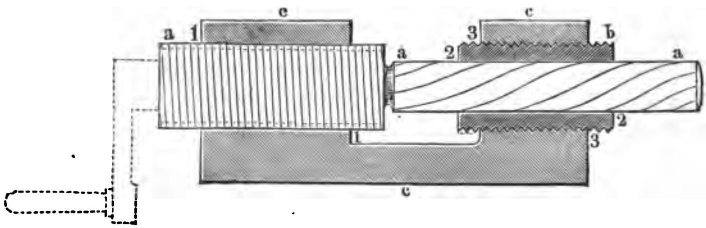


FIG. 267.

Fig. 267 shows another three-link chain, but in this case *all* the pairs are screw pairs. If any one (as c) be fixed, the motions of both the others relatively to it, as well as to each other, are all helical. All the three mechanisms to be obtained from this chain are kinematically the same. The chain, which is given by Reuleaux, does not seem to have found any applications as yet in practical work, for the reason, no doubt, that so few uses exist in machinery for helical motions.

There are a number of mechanisms of a more practical kind in which a single screw-pair is used with a number of other links, very often with the special intention of *fixing*

the mechanism, *i.e.* of preventing any motion of its links except by the action of those forces which cause the screw to rotate. This fixing, as has been already mentioned, is a consequence of frictional resistances, whose nature and

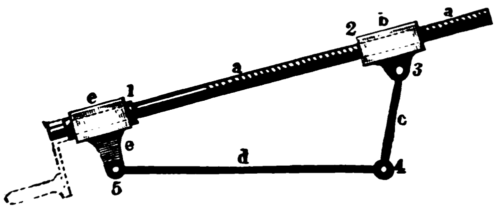


FIG. 268.

magnitude have to be discussed in another chapter. Figs. 268 and 269 show two such chains, the latter an arrangement of Nasmyth's for a dividing machine. In neither case do the problems connected with the chains present any difficulty; the thrust or axial pressure of the screw being

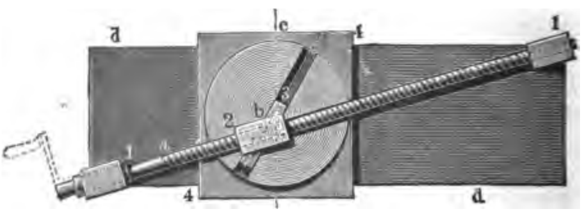


FIG. 269.

once found, it can be taken as a known force, acting on a link of a plane chain, and so dealt with by the methods already discussed.

The steering gear shown in Fig. 270 differs from the other screw mechanisms examined in having a double

thread cut on the screw, so that it can work with two half nuts, one right and one left-handed, moving one forwards, while it simultaneously draws the other backwards. It is, in fact, only a modified form of the mechanism of Fig. 271, where the two screws and nuts are complete and separate.

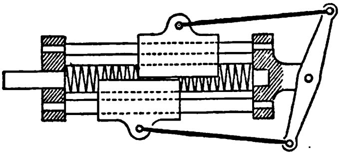


FIG. 270.

The worm and worm-wheel, Fig. 272, form together one of the most familiar combinations containing a screw. We shall see in § 69 that the mechanism shown in the figure is essentially a very special case of screw-wheel gearing, the worm being really a screw-wheel of one, two, &c., teeth, according as it is a single- or double-, &c., threaded. Looking

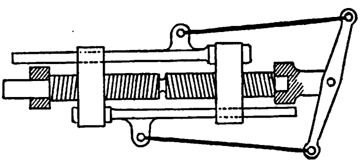


FIG. 271.

at the mechanism at present, however, merely as we have been looking at the other screw-trains mentioned, we see that the pitch circle of the wheel receives from the screw simply the axial component of its motion, exactly as does the link *b* in Fig. 262. The pitch of the teeth of the wheel is the same as the pitch of the screw helix, if it be single-

threaded, or half that pitch if it be double-threaded, and so on. Each complete revolution of the screw therefore carries the wheel round through the angle corresponding to one tooth if the screw be single-threaded, two teeth if it be double-threaded, &c., exactly as if the worm were (as it essentially is) a wheel of one, two, &c., teeth. The mechanical advantage of the combination, for a resistance at a radius

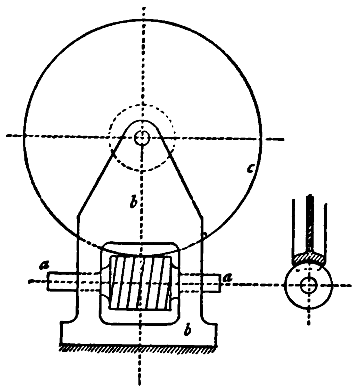


FIG. 272.

equal to that of the wheel, is exactly equal to that of the screw press noticed above, or $\frac{2\pi r}{p}$. But here not only can r be increased, but also the radius at which the resistance acts can often be conveniently made very much less than that of the wheel, as shown, for instance, by the dotted circle. By means of these three links, therefore, a very large mechanical advantage can be gained with the use of very few links and in a very small space.[1] In order that the

[1] As to the relative diameter of the worm and wheel, and of the twist axodes to which they correspond, see § 69.

worm-wheel and worm may gear properly together, it is often assumed that the teeth of the former must be themselves portions of helices having a tangent, in the middle plane of the wheel, coincident with the pitch tangent of the worm-thread. The curvature of such helices being exceedingly small in such a small fraction of their pitch as is represented by the breadth of the worm-wheel, the teeth are usually made straight, and simply inclined at an angle equal to the pitch-angle (or angle whose tangent is $\frac{p}{2\pi r}$) to the position they would occupy in the spur-wheel. If pressure were transmitted from the worm to the wheel always in the middle plane of the latter, this approximation would be reasonably accurate. The point of contact,[1] however, traverses the wheel-tooth from side to side, and (especially if the teeth are hollowed out as sketched to the right of Fig. 272), this causes irregularities of a kind similar to those mentioned on p. 121, in the motion transmitted. This matter has been examined by Professor W. C. Unwin, who has given, in his *Elements of Machine Design*,[2] the only satisfactory investigation of it which we have seen published.

If it is not essential that the axes of the worm and wheel should be at right angles, the arrangement adopted by Mr. Sellers, of Philadelphia, offers many constructive conveniences (Fig. 273). Here the angle between the axes is made less than 90° by an amount exactly equal to the pitch angle of the screw. This turns the worm-wheel into a spur-wheel (a screw-wheel of infinite pitch), its teeth being parallel to its axis. This combination, with reasonably well-formed teeth, runs with exceedingly little friction.

[1] In screw gearing contact occurs at a point on each tooth, not along a line (see § 69).

[2] P. 296, &c., in the fifth edition.

The twist-axis, which takes the place, in such combinations as these, of the virtual axis of rotation of plane mechanisms, as well as of those non-plane mechanisms which are to be examined in §§ 63 to 66, will be found discussed in § 68.

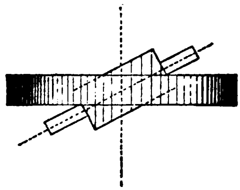


FIG. 273.

§ 63.—CONIC CRANK TRAINS.

IN § 2 we mentioned plane, spheric, and screw motions as the three principal cases with which we had to deal in machinery. Before looking at the more general motions coming under the third head, it will be convenient to examine those of the second. Let PQ, Fig. 274, be a *spheric* section of any body having spheric motion (p. 15), and p and q the paths (on the surface of the sphere) of the points P and Q respectively. We can find a virtual axis for the motion of the body by a method exactly similar to that used for plane motion (p. 40). We can, namely, consider P as moving for the instant along a great circle α touching p in P, and Q along a great circle β, touching q in Q. Drawing great circles α_1 and β_1 normal to α and β, we may say that the instantaneous motion of P is equivalent to a rotation about any diameter of α_1, and that of Q equivalent to a rotation about any diameter of β_1. But α_1 and β_1, being great circles on the same sphere, must have one diameter (here SS_1) in common. The body PQ has therefore for

its instantaneous motion a simple rotation about SS_1, which becomes its *virtual axis*. Taking other positions of P and Q we can obtain other virtual axes, and the locus of these axes will again be an *axode* (p. 52). The axode will here, however, consist of a number of lines all passing through the same point O, the centre of the sphere, so that it will be a cone instead of (as for plane motion) a cylinder. In

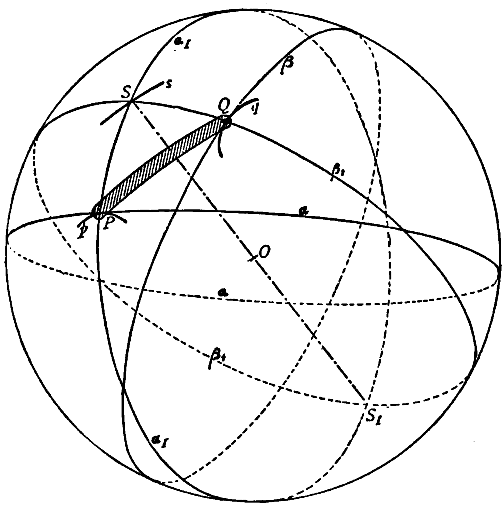


FIG. 274.

general, of course, the cone will be non-circular, just as the cylindrical axode was non-circular, except in the special cases where the centrodes were circular, as on pages 119 and 146. The curve s, which is the locus of the points S, the intersections of the virtual axes with the spheric surface, has some of the properties of the centrode. For the relative (spheric) motions of any two bodies, for instance, measured

on the same sphere, there are two such curves, which touch each other always in one point, and that point is the point S, which, along with the centre O, determines the position of the virtual axis.[1] The two curves *roll* upon one another, as the bodies to which they correspond move, exactly as do the centrodes in plane motion. But we cannot speak of such a point as S as a virtual centre, for the different points in PQ are not points in a plane passing through S, and their virtual motions are not rotations about any one such point, but about different points in the line SO. The motion, therefore, when reduced to its lowest terms, is a rotation about an axis, for which axis a point can *not* be substituted, as formerly in § 7.

Let a, b, and c, be any three bodies each having spheric motion relatively to the other. For these motions there will be three virtual axes, which we may call A_{ab}, A_{ac}, and A_{bc} respectively. The theorem of the three virtual centres (p. 73) is here represented by a theorem as to these axes, which may be stated thus: **If any three bodies, a, b, and c, have spheric motion, their three virtual axes, A_{ab}, A_{ac}, and A_{bc}, are three lines in one plane.** We have already seen that these three lines must all pass through one point, O. The simplest proof of this theorem is one corresponding to that formerly given, namely, the following:[2] The line A_{bc} is a line belonging to both the

[1] As the same construction gives us the points S and S_1 simultaneously, and as there is no kinematic difference between them, we may indeed say that each curve is a double one, having two similar and equal parts placed oppositely on the sphere. But as these two parts are precisely similar and equal, and as either one of them by itself, along with the given centre O of the sphere, is sufficient to determine the axode, it is unnecessary to trouble ourselves to consider more than the one curve s, or point S, which happens in any construction to be the more convenient.

[2] If a figure would make it easier for the student to follow this statement, Fig. 30 can be used, taking the paper as a projection of a spherical

bodies b and c. As a line in the former it is turning, relatively to a, about A_{ab}. It must therefore be moving in a plane at right angles to the plane containing the lines A_{ab} and A_{bc}. As a line in c, it is turning, relatively to a, about A_{ac}. It must therefore be moving in a plane at right angles to the plane containing the lines A_{ac} and A_{bc}. It can be moving only in one direction at one time, so that two planes which both pass through it, and are both normal to that direction, must coincide. The planes A_{ab} A_{bc}, and

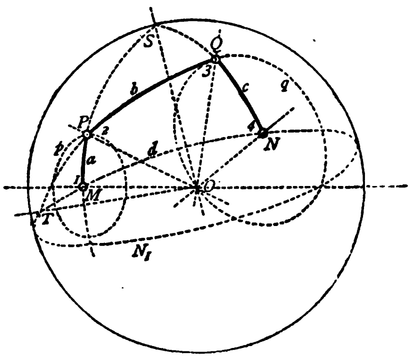


FIG. 275.

A_{ac} A_{bc}, therefore coincide, and the three lines A_{ab}, A_{ac}, and A_{bc} are three lines in one plane. We shall find considerable use for this theorem later on.

Referring again to Fig. 274, it will be noticed that the paths p and q were assumed quite arbitrarily. They may be, for instance, circles on the surface of the sphere, as sketched in Fig. 275. In that case they might be constrained by the use of links MP and NQ, pivoted at M and N

surface, and the points O_{ab}, &c., as the traces on that surface of the axes A_{ab}, &c.

respectively on axes passing through *O*. If the sphere carrying these axes be supposed fixed, the three links *MP*, *PQ*, and *QN* form along with it a four-link mechanism in which the motions are completely constrained. But the *shape* of the links is immaterial (p. 66), so that we may omit the sphere itself, connect *M* and *N* by a bar as in the other cases, and we get the four-link chain shown in Fig. 276. Here the essential matter is that the four axes of the pairs of elements should all pass through the same point *O*. In the case of plane motion the axes were parallel, the point

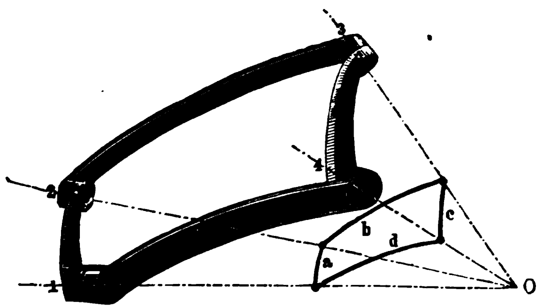


FIG. 276.

O was infinitely distant. In the case of spheric motion, then, we may consider that we have simply brought the point of intersection of the axes nearer, without changing anything else. Comparing the chain here shown with that of Fig. 25, p. 61, it will be seen that it is altered in no other respect. It still contains four links, connected by four turning pairs, and the relative lengths of the links are nearly the same.[1] In both cases, if *d* be fixed, *c* will

[1] In this and following figures the links are shown as curved bars. It is hardly necessary to explain that this is done only to make the figure

swing and *a* rotate, so that the new mechanism may be looked at as simply the old one bent round, so as to bring the point of intersection of its four axes to some near position. For this reason we can still call such a mechanism a lever-crank, but we shall call it a **conic lever-crank**, to distinguish it from the former plane one.

Some characteristic points about these conic mechanisms require notice before we proceed to examine them. In the first place the relative *lengths* of the links are no longer matters of simple linear measurement (for the actual constructive links are not lines on the surface of a sphere), but depend on the *angles* respectively subtended by them. Moreover, no link can subtend an angle greater than a right angle. For if MON, Fig. 275, had been greater than 90°, we could have used N_1 instead of N, and MON_1 would have been less than 90°. We have the same possibility for every link, for we have already seen that to every point, P, Q, &c., there corresponds another on the opposite side of the sphere, having an exactly similar path (p. 490). Hence every link may be said to have either of two angular lengths, as a and $180°—a$, one of which is the supplement of the other. The motions are not affected by which of these lengths are used, but to avoid confusion in speaking of them it is generally convenient to state the length which is less than 90°. The constructive appearance of the mechanism, on the other hand, is so much changed by such alterations as often to place considerable initial difficulties in the way of identifying or understanding it. Thus it is at first sight difficult to recognise Fig. 277, and still more Fig. 278, as being mechanisms not merely similar to,

more intelligible. The links may, just as before, be of any convenient shape or dimensions, straight or curved, so long as only the axes of the elements occupy their proper positions.

but absolutely identical with, that shown in Fig. 275. But examination will show that no change whatever has been made, except the substitution of links subtending the supplementary angles, (in Fig. 277 + 180°) as just mentioned.

In Fig. 279 a conic mechanism is shown, in which two links, *c* and *d*, are made each to subtend a right angle. The constructive form of *d* is made different from that of *c*, that the motions may be realised more easily, but there is no kinematic difference between them. It will be noticed that if *d* be fixed we have a mechanism in which the point

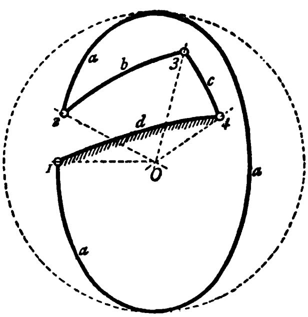


FIG. 277.

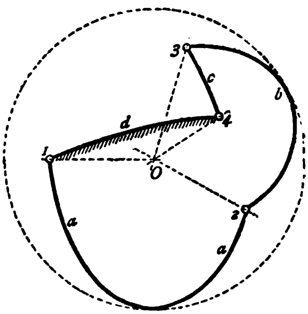


FIG. 278.

3 moves always in a great circle at right angles to one of the axes of *d* (the vertical one in the figure) and in the plane of the other (the horizontal one in the figure). The link *a* rotates as before, and the link *c* still swings or reciprocates, its end-point 3 moving always in the same plane, as we have just seen. The motion of the link *b* corresponds exactly, in consequence, to that of the connecting-rod in the ordinary slider-crank chain (Fig. 26); it might be described accurately enough as a connecting-rod working round a corner!

By expanding[1] the pair 4 in Fig. 279, and bringing it down to the level of the plane 31, the link *b* becomes externally a slider working in a curved slot in *d* (Fig. 280), and the identity of the mechanism with the slider-crank becomes obvious, even on the surface. It may be said to be simply a slider-crank bent round, exactly as in the former case we had a lever-crank bent round. The right-angled links of the conic train correspond to the infinite links of the plane train; motion along a great circle corresponds to motion along a straight line.

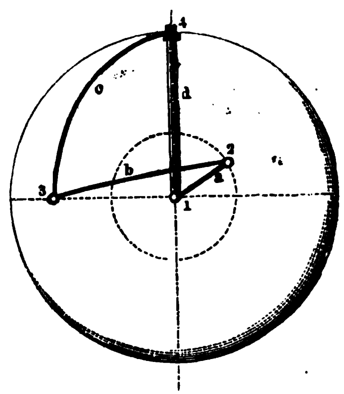


FIG. 279.

The positions of the virtual axes of the links in a conic chain correspond exactly to their positions in the plane chains. For clearly if the points *P* and *Q* (Fig. 274) are constrained in their motion by two rotating or oscillating links, the lines α_1 and β_1, at right angles to the point-paths, must be the centre lines of those links, and the point *S*, which fixes the virtual axis of *PQ*, lies at their join.

[1] See § 52.

Hence, exactly as in the case of plane chains, **the virtual axis**[1] **of either pair of opposite links is the join of the planes of the other two:**[2] **the virtual axis for any pair of adjacent links is the join of their own planes, and is a permanent axis.** Thus in Fig. 281, *SO* is the virtual axis for *b* and *d*, and *TO* for *a* and *c*, and for adjacent links, 1*O* is the virtual axis for *a* and *d*, 2*O* for *b* and *a*, &c. The same lettering is used in

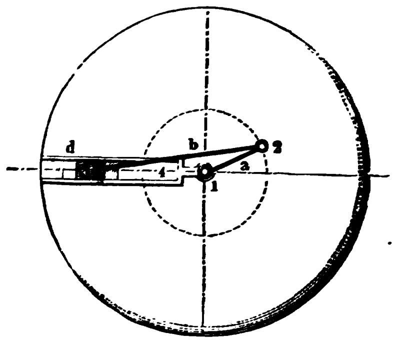


FIG. 280.

Fig. 282, to show how these axes come in the conic slider-crank. It will be there seen how the linear velocities of the points 2 and 3 will be equal when $S2 = S3$, and how *SO* will coincide with 4*O* when the crank is in its mid-position. In this position *S*2 will be at its shortest ($=90° - a$) and *S*3 at its longest ($=90°$); therefore for this position of the mechanism (crank in mid-position) there

[1] We have already stated the reasons which compel us here to state the proposition thus instead of with the use of the word *centre*, as formerly.

[2] Compare the theorem on p. 72 and the construction shown in Figs. 29 and 31.

will be the maximum value of the ratio $\frac{\text{vel. } 3}{\text{vel. } 2}$, which corresponds to the ratio $\frac{\text{vel. crosshead}}{\text{vel. crank-pin}}$ in an ordinary engine. There will be two positions, one on each side of the middle, when $S2$ and $S3$ are equal, and where therefore vel. 2 = vel. 3. If the link *b* subtends a right angle as well as the links *c* and *d*, these two positions will be symmetrical to the mid-position, otherwise they will be unsymmetrical.

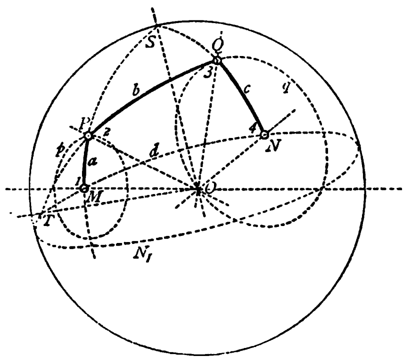


FIG. 281.

As the virtual axes can be found in this way for any conic mechanism, all the kinematic and kinetic problems which we could solve by their help in connection with plane mechanisms can be similarly solved here. Such additional difficulties as there are arise chiefly from the difficulty of recognising the mechanisms under most elaborate constructive disguises, and from the unavoidable necessity of drawing ellipses where formerly straight lines were sufficient. We shall now proceed to examine two or three of the principal practical applications of conic chains,

and to work out in connection with them problems similar to those which we have already solved with plane mechanisms. The methods of treatment and solution given in the next two sections will be such as are applicable to any conic trains, although actually applied here only to special ones.

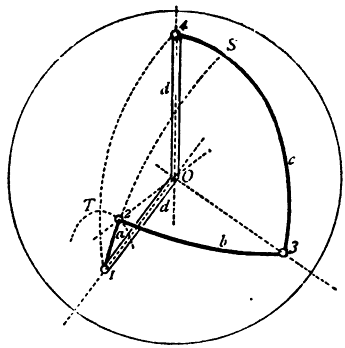


FIG. 282.

§ 64.—THE "UNIVERSAL JOINT."

PROBABLY the most familiar example of a conic crank train occurring in practical work is the Hooke's coupling or **Universal joint**, shown in Figs. 283 to 285, of which the first shows the joint in a form more or less resembling that commonly used in construction, while the two others show it in the schematic form adopted in the last section, and show the link *a* alternatively made to subtend an acute angle and its supplement. Corresponding links and pairs are noted by the same letters or numbers in the two figures. The chain consists of four links connected by turning pairs, whose axes all meet in one point *O*. Three of the links—

b, *c*, and *d*—subtend each a right angle, the fourth, *a*, the fixed link, subtends some much larger or (p. 493) smaller angle.

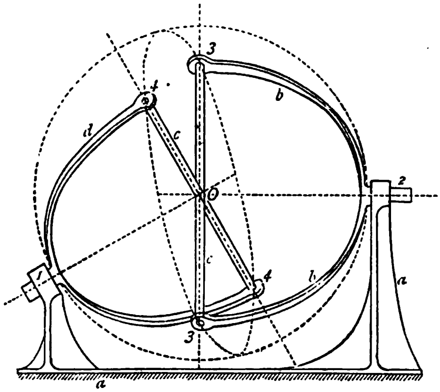


Fig. 283.

In Fig. 283, the pairs 1 and 2 of *a* are placed, as they are usually in construction, on opposite sides of the mechanism.

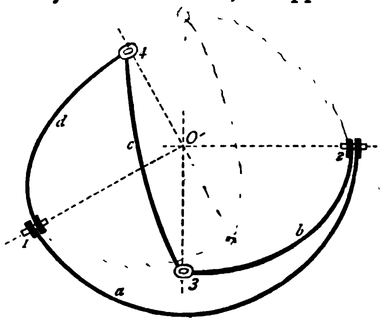


Fig. 284.

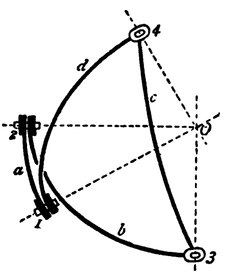


Fig. 285.

In Fig. 285 they are shown on the same side, and it will be seen from this sketch that the mechanism is shown in a

position corresponding to those on the lower side of Fig. 55 (p. 111). We have already seen that the change from an angle to its supplement is without influence on the motions occurring in the mechanism, and does not, therefore, constitute any real difference between them. The links *b* and *d* are little altered in appearance, but in Fig. 283, for the sake of securing additional steadiness in the mechanism, *both* ends of 3 and 4 are paired to *c*, and not only one, as in Figs. 284 and 285. Lastly, the link *c* is transformed (like *d* in Fig. 282) into the shape of a cross, paired at 3 and 4 with the double-sided forks of *b* and *d*. Kinematically it is equally a link carrying two elements of turning pairs (pins or eyes), with their axes at right angles to each other, whether it have the form shown in Fig. 284 or be made as the cross of Fig. 283.

If we take an ordinary lever-crank, and fix the short link *a* (as in Fig. 55, p. 111, for instance), we obtain a mechanism commonly known as a "drag-link coupling." The links *b* and *d* revolve on parallel axes, the one driving the other through the "drag-link" *c*. If the mechanism is a parallelogram (Fig. 43), so that $c=a$ and $b=d$, the two revolving links are turning always with the same angular velocity, but the mechanism has two change-points (p. 147). With the ordinary proportions of links, *b* and *d* turn with constantly varying velocity ratio, as we have seen in connection with Fig. 55. When the lever-crank is turned into a slider-crank, and the link *a* again fixed, we get the "quick-return" mechanism of Fig. 126. Here again *b* and *d* revolve, and the one drives the other through the link *c* with constantly varying angular velocity ratio. When, lastly, the lever-crank is turned into a conic chain, and the link *a* fixed, we obtain the mechanism before us, which moves in precisely the same fashion, and is similarly used for the transmission of rotation from one shaft to another. The

links *b* and *d* rotate, and the one drives the other through the intervention of the link *c*. Here, again, both driving and driven shafts have their bearings in *a*, the fixed link, but they are now angled to each other instead of being parallel. The angular velocity ratio transmitted is a constantly varying one. The Hooke's joint is in effect a drag-link coupling between shafts whose axes are not parallel, but meet in a point at a finite distance.

It follows from the construction of the mechanism that the planes of *b* and *d* are at right angles to each other four times in each revolution (this can be seen at once from the figures following), and at these instants the two shafts are revolving with the same velocity. The links *b* and *d* thus make quarter-revolutions in the same time. Between these positions the angular velocity ratio varies very much, and varies the more the greater the angle between the shafts. This variation is so great as to make the mechanism practically unusable unless the angle subtended by the link *a* (*i.e.* the angle between the shafts) be small; while if *a* be made $=90°$, the mechanism ceases to be moveable at all.

The first problem in connection with the universal joint is the finding of the position of *d* for any given position of *b* (we may suppose *b* the driving and *d* the driven shaft), the next problem is to find their relative velocities in any given position, and the last to find the corresponding relations between effort and resistance.

Before going on with these problems it is necessary to find where the virtual axes of the mechanism lie. They are shown in Fig. 286, in which four planes are drawn instead of the four bars, in order to make the intersections more clearly visible. The planes are placed exactly in the position of the links of the three last figures, the actual position of the links of Fig. 285 being duplicated by dark

lines on the edges of the planes. The plane of the link a, the vertical circle, is taken as coinciding with the plane of the paper. The three axes A_{ab}, A_{ad}, and A_{bd} (see p. 490) are therefore three lines in the plane of the paper. A_{ab} and A_{ad} are the axes of the shafts, and are therefore known. A_{bd} is the join of the planes of the cross (the link c) and the shafts (the link a). The axes A_{bc}, A_{cd}, and A_{bd} are again all in one plane. The two first-named are the arms of the cross, the last we have just found. The plane containing

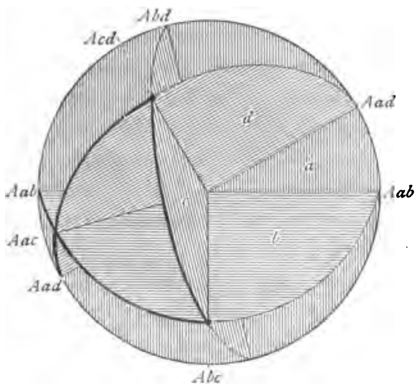


FIG. 286.

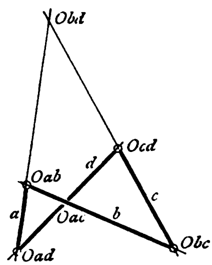


FIG. 287.

them is, of course, the plane of the cross. The axis of the cross c relatively to the fixed link a, A_{ac}, is the only one the drawing of which offers some difficulty. Its position is, as we know, the join of the planes of b and d, the two forks. A_{ac} lies along with A_{ab} and A_{bc} in the plane of the fork b, and along with A_{ad} and A_{cd} in the plane of the fork d. For the sake of showing how completely the same the whole matter is with that which we examined in connection with plane motion, the corresponding plane mechanism is

drawn in Fig. 287, in corresponding position to Fig. 286, and its six virtual centres marked.

A most obvious difference between the plane and the spheric mechanisms is felt as soon as any attempt is made to handle them on paper. Any number of positions of the plane mechanism could be drawn at once, without the slightest difficulty, as soon as ever the lengths of the links were given. With the spheric mechanism, on the other hand, although it is easy to draw the chain in the position of Fig. 288 and in a position 90° from it, to draw it in any other position is not so simple a matter, and, indeed, cannot be done without projection in two planes. This does not indicate in any way that these mechanisms belong to a higher order (§ 59), but follows simply from the fact that the non-plane motions cannot be fully represented on paper without projection in two planes.

We shall now proceed with our first problem: the finding of the position of the whole mechanism when that of any one link is given. The link *b*, which we may consider as a driving shaft, is generally the one whose position is given. We shall first see how to find corresponding positions of the links *d* and *c*, the former being the more important.

In Fig. 288 is shown a Universal joint drawn, as before, so that the plane of the paper coincides with that of the two shafts, that is, of the link *a*. The fork *b* starts from a vertical position, the plane of *d* is at right angles to the vertical plane. In the side view, Fig. 289, the two axes of the link *c* appear vertical and horizontal, as OM and ON. The ellipse *n*, which is the projection (in Fig. 289) of the path of the point N (or 4), must be drawn first. Let the link *b* move through any angle, β, so that OM takes up the position OM_1. The arm ON must always *appear*, in Fig. 289, to be at right angles to OM, and the projected path

of N has already been drawn, so that the point N_1 can be found at once. N_1ON (Fig. 289) is the projection only, on the plane of the paper, of the real angle moved through by ON (*i.e.* by the fork d) in a plane inclined at an angle θ to the plane of the paper. The real angle is therefore to be found by turning back the one plane upon the other, which is done if we turn back the ellipse M_3N_3N until it coincides with the circle M_3MN. The point N_1 then comes to N_2, so that N_2ON, or α, is the real angle moved through by d, while b is turning through the angle β.

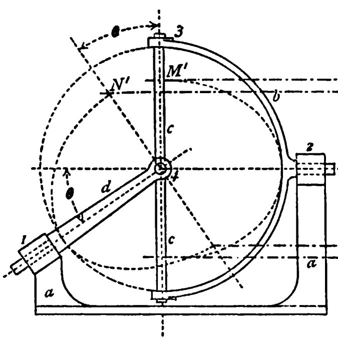


FIG. 288.

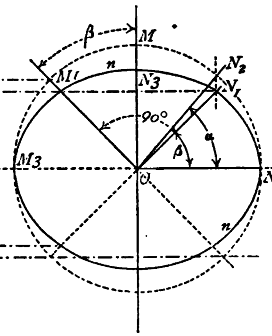


FIG. 289.

The point N_2 is always the point of the circle beyond N_1, and the angle NON_1 is always equal to the angle β. We have thus solved the first part of our problem, the finding of the position of one shaft when that of the other is given. It can easily be seen from the construction that (as has been already mentioned) the two shafts make quarter turns in the same time, although their velocities vary very much within each quadrant. Thus, when OM reaches OM_3, the projection of the position of ON will be ON_3, and the real

angle, as well as the projected angle, turned through by ON will be a right angle.

Algebraically, using the symbols of the last two figures,

$$\sin \alpha = \sin \beta \frac{ON_1}{ON_2 \cos \theta} = \sin \beta \frac{ON_1}{ON \cos \theta}.$$

If we wish not only to find the position of d, but also to draw the mechanism in its new position, we may proceed as follows (Figs. 290 and 291):—we know that in the projection, Fig. 290, the one arm of the cross must always appear to lie along the line m and the other along the line n, the one

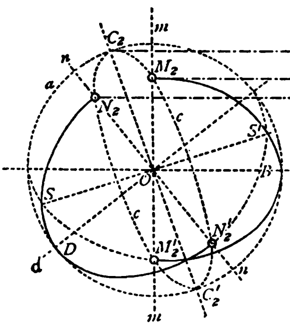


FIG. 290.

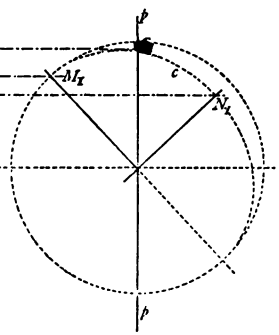


FIG. 291.

at right angles to the axis of b and the other at right angles to the axis of d. We can therefore project the points M_1 and N_1 from Fig. 291 at once on to these lines, at M_2 and N_2. (The other ends of the cross arms can of course be marked at once, at points equidistant from O, at M'_2 and N'_2). The fork of b will then appear as part of an ellipse, whose semi-axes are OM_2 and OB, and the fork of d as an ellipse whose semi-axes are ON_2 and OD; they are so shown in the figure.

We have thus been able to draw the links of the

mechanism in their right relative positions. Before we can go further we must, as we know, find the six virtual axes which belong to the mechanism. Four of these we have already found in Fig. 290, the four corresponding to the four pairs of adjacent links, the lines *b*, *d*, *m*, and *n* in the figure. The first two lie in the plane of the paper, the last two are inclined to it at angles made determinate by Fig. 289. The two axes of opposite links, A_{bd} and A_{ac}, have still to be found. The first of these is at the join of the planes of *c* and *a*, as shown in Fig. 286. It is therefore a line in the plane of the paper in Fig. 290, and in a plane (as *p*) at right angles to the paper in Fig. 291. It is probably most easily found by taking both *a* and *c* as circular plane figures, and finding their intersection. The projection of *c* in Fig. 291 will be an ellipse whose semi-axes are OM_1 and ON_1. It is only necessary to draw as much of this ellipse as will give us the point C_1, where it cuts the plane *p*. This point can then be projected to the *a* circle in Fig. 290, and so gives us C_2, the line $C_2OC'_2$ being the line we require, the intersection of the planes of *c* and *a* of the cross and the shafts. (This line is also the axis of the *c* ellipse sketched in Fig. 290, and formerly in Fig. 286, which passes through the four points, N_2, M'_2, N'_2, and M_2, and has, like the others, the point O for its centre.)

The virtual axis of *a* relatively to *c* is the join of the planes of the two forks *b* and *d*, as shown in Fig. 286. It can here be easily found, without the aid of the second view, by drawing the fork ellipses until they cut each other (as both are projections of great circles on the same sphere they *must* cut somewhere) in the point S. The line SOS' is the required axis A_{ac}.

Having now the means of drawing the virtual axes as well as the links of the mechanism, we can proceed to problems con-

nected with the relative velocities of its different links. Angular velocities are here of so much greater importance than linear velocities that we may confine ourselves to them. The method to be used for finding the relative angular velocities of two links —say *b* and *d*, the two shafts—is essentially identical with that employed in the similar case with plane motion which was illustrated in Fig. 45, p. 99. We require, first, the three virtual axes, A_{ab}, A_{ad}, and A_{bd} (*a* being the fixed link), which we have just found, and which we know to be three lines in the plane of the paper in Figs. 288 or 290. The axis A_{bd} being a line common to the two bodies *b* and *d*, *any point in it* must have the same linear velocity relatively to *a*, whichever body it may be considered to belong to. But each body is turning, relatively to *a*, about a known fixed axis, A_{ab} for *b*, and A_{ad} for *d*. The angular velocities of the bodies are therefore inversely proportional to the radii, measured from these axes, of any one point in their common line. The similar case for plane motion was stated on pp. 96 and 97, § 15. We then required to find three virtual centres—the fixed point of each link and the common point of the two links—and these three points were in one line. Here we require, similarly, to find three virtual axes—the fixed axis of each link and their common axis—and these three lines are in one plane. Formerly we said that the virtual centre of the two moving bodies was a point which had the same linear velocity in each. Now we can say that the virtual axis of the two moving bodies is a line all of whose points have the same linear velocities in each.[1] The necessary constructions are therefore essentially the same as formerly, differing only because projection on two planes

[1] This also might, of course, have been equally said before, but for reasons which were explained we preferred to express it in the former fashion.

is here required, while formerly everything could be shown upon one plane only.

Having found the three lines A_{ab}, A_{ad} and A_{bd} as in Fig. 292, which simply duplicate the axes found in Fig. 290, we have only to measure the distances of any point, as C, in A_{bd} from A_{ab} and A_{ad}; these distances being respectively CB and CD in the figure. They can be measured directly, as all the lines with which we are here concerned are lines in the plane of the paper. The angular velocities of the links

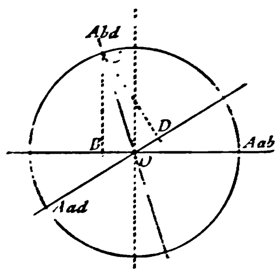


FIG. 292.

b and d are, therefore, inversely proportional to the distances CB and CD, that is,

$$\frac{\text{angular vel. } b}{\text{angular vel. } d} = \frac{CD}{CB},$$

Algebraically this ratio is equal to

$$\frac{\tan \alpha + \cot \alpha}{\tan \beta + \cot \beta},$$

using the nomenclature for the angles employed in Fig. 289 above. The limiting values of this ratio are $\cos \theta$ when OM (Fig. 289) is in the plane of a, and $\frac{1}{\cos \theta}$ when ON is in the plane of a, the angle θ being the angle between the shafts.

At one position within each quadrant the two shafts have for an instant the same velocity,[1] and as we have already seen, they make quarter-revolutions in the same time, so that the *mean* value of their angular velocity ratio is unity.

Were it desired to find the relative angular velocities of c and one of the other links, say b, a problem which may occasionally occur in some modern machines, it can be found in a precisely similar manner. First, the three axes A_{ab}, A_{ac}, and A_{bc} are found; they all lie (see Fig. 286) in the plane of the fork b. Then the distances are measured from any point in A_{bc} to the other two axes, and the calculation made exactly as in the last case. The only additional complication is that the plane of b is not, as we have drawn the mechanism, the plane of the paper, and has to be turned down into it, about A_{ab} (exactly as we did on p. 504), before the distances can be measured.[2] A figure drawn like Fig. 286 will generally make the position of the virtual axes quite clear and intelligible, although their position may be a little difficult to realise in looking at the mechanism itself; it may often be worth while, therefore, in working practical problems with these mechanisms, to use such a figure for reference.

The only problems which now remain to be considered respecting the class of mechanisms of which we have been taking the universal joint as representative, are those involving the static or kinetic balance of forces in them. The general problem which was treated for plane motion in § 40, the finding of the force which must act in a given direction at *any* point of any link in order to balance a given force acting at any point in any other link, can be solved here by the same general and direct method which was employed

[1] See the curves of velocities in Fig. 307, § 65.
[2] An example of this calculation occurs in § 65.

before. The application of this method is practically troublesome, however, because of the amount of projection it involves. It is more convenient, and for many purposes sufficient, to suppose all forces to be acting through the two moving virtual axes in the mechanism which are also permanent axes, namely (if a be the fixed link) the axes A_{bc} and A_{cd}. In order to do this, it is only necessary to divide the magnitude of each force by the alteration of radius. This will be clear from Fig. 293, where, for the sake of

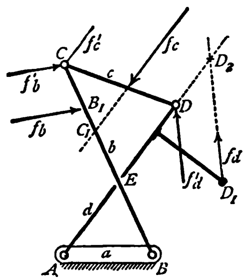


FIG. 293.

clearness, a plane mechanism is drawn. Instead of taking the force f_b at B_1, it is shifted parallel to itself to C (which is here O_{bc}), and is taken as $f'_b = f_b \times \frac{BB_1}{BC}$. Similarly f_d is taken as acting at D ($= O_{cd}$) instead of at D_1, and its magnitude is supposed changed to $f'_d = f_d \times \frac{AD_2}{AD}$. Forces acting on c at any other points than C or D can in the same way be reduced to either of those points; thus f_c may be replaced by $f'_c = f_c \times \frac{C_1E}{CE}$. It is generally most convenient to perform this reduction arithmetically. It is not always

quite so easy to see what the distances BB_1, BC, &c., are in the conic mechanisms as in the plane, and Fig. 294 is therefore drawn to correspond to the last one, but for a conic train. Here a force f_d acting on d in any direction at D_1, may be replaced by a force f'_d, acting in a parallel direction at D, and equal to $f_d \times \frac{D_1A}{DO}$. Similarly any force f_b acting on b in any direction at B_1 may be replaced by a force f'_b acting in a parallel direction at C, and equal to $f_b \times \frac{B_1B}{CO}$. The case is not quite so simple here with forces on the link c.

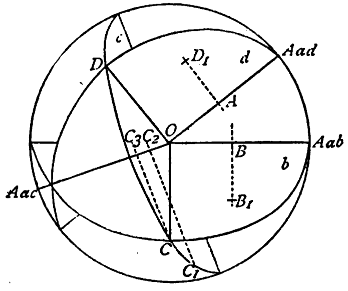


FIG. 294.

Any force f_c at C_1 may be replaced by a force f'_c acting at C, and equal to $f_c \times \frac{C_1C_2}{CC_3}$. But the radii C_1C_2 and CC_3 are not parallel (although their projections are), so that these distances cannot be measured on the paper, their real lengths must be found by projection. Further, the new force f'_c cannot be taken as parallel to f_c, but only as making the same angle to OC that f_c made to OC_1. In this case, as also in cases where such points as B_1 or D_1 do not lie actually in the planes of the forks, it is more convenient

to treat the force in the way described in the following paragraphs, before transferring it to another point.

In dealing with plane mechanisms, we assumed that the forces in action were always in the plane of the mechanism. Any components which they had normal to that plane were balanced by the profiles (p. 54) of the elements, which rendered impossible all motions not parallel to the plane. In the plane itself we virtually resolved the force at any point into components parallel and normal to the direction in which the point was moving. The parallel component had to be balanced by other external force or forces, the normal component was balanced by the stresses in the links. Considering the force as acting on a body at some particular point, its tendency to turn the body about its virtual centre, or *moment* about the virtual centre (p. 269), was equal to the product of its component parallel to the direction of motion of the point and the virtual radius of the point. If the question were merely of the moment of the force upon the body, without reference to its action at any particular point in the force line, we saw that it was unnecessary to resolve the force in its own plane. The moment was simply equal to the whole magnitude of the force multiplied by the virtual radius of the force line—that is, its perpendicular distance from the virtual centre.

These simple matters have been repeated here that they may be the more distinctly before us in their applications to spheric motion, with which we have now to deal. The links in a conic mechanism have at each instant just the same sort of motion as those of a plane mechanism—each link, namely, is in rotation about a determinate axis. But these axes, so far as they are non-permanent or instantaneous, are continually shifting their *direction* as well as their position. The direction of the plane normal to the

axis—or plane of effective force, as we may call it—is therefore continually changing, and even at any one instant, as the axes of the different links are not parallel, the planes for the different links are not parallel, instead of being, as before, all coincident. This difference introduces some new complexity into the problems; but if it only be borne in mind, the whole of the last paragraph may be taken as applying equally to spheric and to plane mechanisms. Two or three illustrations may be given here; they will serve to show the essential identity of the problems as well as to illustrate their superficial differences.

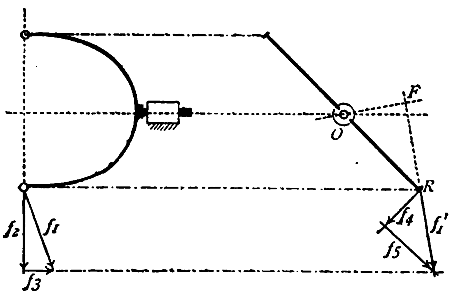


FIG. 295. FIG. 296.

A force f_1, given by two projections in Figs. 295 and 296, acts on one of the forks of a universal joint. Resolving into f_2 and f_3 in Fig. 295, we find at once f_2 as the projection of the nett turning force, and f_3 as the component in the direction of the axis. The latter causes axial pressure or thrust in the bearings, but causes no motion of the fork, and is therefore negligible so far as motions are concerned. The plane of the paper in the second view, Fig. 296, is a plane at right angles to the axis, and is therefore the plane

in which effective forces act. The projection of f_1 in this plane, namely f'_1, gives the real length of f_2. If our object is merely to find the turning moment, we have it at once as $f'_1 \times OF$. But if we wish to know more completely what is occurring, we must resolve f'_1 in the direction of motion of the point R, at which it really acts, and normal to that direction, *i.e.*, into f_4 and f_5. The effective turning moment is then $f_4 \times OR$ (which is, of course, $= f'_1 \times OF$), and the component f_5 is the magnitude of the side pressure in the bearing.

We get nothing essentially different from this if the given force act upon the cross (the link c, Fig. 283) instead of upon b or d. We have only to choose the position of the planes of projection so that they bear the same relation to the virtual axis of c (A_{ac}, Fig. 286) as the planes of Figs. 295 and 296 bear to the axis of b, A_{ab}. One of them, namely, should contain the axis, and the other be at right angles to it.

It will be seen that in neither case have we done more than, or differently from, what we should have done with a simple "turning pair," if only the force in action had not lain in a plane at right angles to its axis.

We may now assume that we have to deal only with forces resolved in the direction of motion of the points on which they act, and, further (as we have always tacitly assumed in connection with plane motion), that the force so resolved may be taken as acting in any plane normal to its virtual axis,[1] so that it may be shifted, if necessary, parallel to the virtual axis to any extent. With these assumptions we can proceed to look at such cases of the

[1] Forces acting on different points of the link may, therefore, first be each resolved in planes at right angles to the virtual axis and normal to such planes, and then the resolved parts in the parallel planes may be all added together or otherwise treated, so far as motion is concerned, as if they were all in one plane.

static balance of forces as we considered in §§ 38 to 41 for plane mechanisms. For the balance of forces on any one link we may either simply balance moments, as on p. 285, or may proceed by resolution through the virtual axis, as on p. 280, whichever happens to be most convenient. It is not necessary to give any example of this. For the balance of forces on different links, whether adjacent or non-adjacent, it is frequently most simple to determine the relative angular velocities of the links, and to calculate the balanced forces from these, remembering that the turning moments on any links which are in static equilibrium must have magnitudes inversely proportional to the angular velocities of the links. Suppose, for example, that b and d, the two shafts, are the two links concerned, and that their angular velocities for the instant are v_b and v_d respectively, the ratio between those quantities being found as on p. 508. Let f_b be the resolved part (as f_4 in Fig. 296) of the forces acting on b, and tending to cause its rotation, and r_b be the radius of f_b (as OR in Fig. 296), then (using similar lettering for forces, &c., on d) we must have

$$\frac{M_d}{M_b} = \frac{r_d f_d}{r_b f_b} = \frac{v_b}{v_d}$$

and $f_d = f_b \cdot \dfrac{v_b}{v_d} \cdot \dfrac{r_b}{r_d}$.

The force f_d thus calculated is the force which, acting in a plane normal to A_{ad}, the virtual axis of d, and at a distance r_d from that axis, will balance the force f_b acting in exactly similar fashion on b. If the real force on d is not acting in the plane mentioned, but at some angle θ to it, the whole magnitude of the force must be $\dfrac{f_d}{\cos\theta}$, while it will have a component equal to $f_d \tan\theta$ acting along the direction of the axis A_{ad}, and causing axial thrust in the bearings of the mechanism.

For the purpose of finding the variation of balanced resistance and making a diagram such as Fig. 146, p. 321, as we shall presently do, this method is all that is required. But for designing a machine it may be insufficient,—we may require to work through the intermediate link *c* in order to find the stresses to which it is subject in transmitting the driving effort from *b* to *d*. Suppose, for example, that a known force acts at the point 3, Fig. 288, and in the direction of its motion, let it be required to find the force in the line 43 necessary to balance it, which would, of course, be the stress in a straight bar connecting-rod coupling those

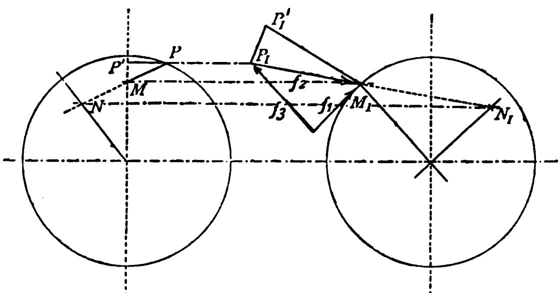


FIG. 297. FIG. 298.

points. In Figs. 297 and 298, the points M and M_1 are the two projections of 3, and N and N_1 the two projections of 4, their positions corresponding to those before determined and sketched in Fig. 288, and the planes of projection being arranged as before. The given force on M (in the plane of Fig. 298) is f_1, and this force can be resolved (in that plane) into f_2 in the given direction N_1M_1, and f_3 through the virtual axis. f_3 is balanced by the pressure of the shaft against its bearings. f_2, or P_1M_1, is the projection, in the given plane, of the force required to balance f_1. In Fig. 297

PM is the projection of the same force. The real length of this force can be found by turning it down in the usual way, namely, by setting off $P_1P'_1$ equal to PP'. The real value of the pressure acting along the line *NM* is thus found to be $M_1P'_1$.

In § 16, Figs. 55 and 56, we gave a diagram of relative angular velocities of an ordinary drag-link coupling, and in the same section, p. 113, it was pointed out that the chain of Figs. 32 or 53, with the link *a* fixed, gave also a coupling ("Oldham's") for parallel shafts, but one which transmitted a constant velocity ratio equal to unity, instead of a varying one. That mechanism was one with three infinitely long links (based on the slider-crank with infinitely long connecting-rod, §§ 12 and 52), and was one, therefore, corresponding exactly to the Hooke's joint, with its three right angled links. Turned into a conic train, however, the velocity ratio transmitted is no longer constant, although (as also with the mechanism of Fig. 55) its *mean* value is unity. Its value varies very greatly at different parts of the stroke. A diagram of velocities will be found in the next section in connection with the engines based upon this mechanism, which are there examined.

§ 65.—DISC ENGINES.

IT is now some sixty years since some inventive mind, longing for novelty, found out that any such conic chain as we have examined in § 63 could be made available as a steam engine. From the form then given to the link *b* the engine was called a **disc engine**, and this name has been subsequently applied generically to the large number of steam engines which have been founded on conic mechanisms. The real nature of these mechanisms was practically

unrecognisable until the kinematic analysis of Reuleaux appeared, and Reuleaux himself was the first to point out its application to them, and their essential identity with certain familiar plane mechanisms—ground over which we have followed him in §§ 63 and 64. In his book[1] he examines and analyses a number of those forms of disc engines which had been proposed previous to its publication, and there is no need that we should go over the same ground. None of the forms which he describes were such as to give any promise of practical success, and probably not a single one of these older forms still survives. Lately however, some other forms of disc engine have been devised, in which the one advantage of such engines, the high speed, has been made the most of, and the several disadvantages have been, by careful design, much reduced. Of these recent disc engines we shall examine in this section the two latest, each of which appears to have possible future practical usefulness, for high speed driving, before it. One is the invention of Mr. Beauchamp Tower. It has been already used to a considerable extent for electric lighting purposes. It will be found described in a paper by Mr. R. H. Heenan, published in the *Proceedings of the Institution of Mechanical Engineers* for 1885. The second is the invention of Mr. John Fielding—it has appeared for the first time at the International Inventions Exhibition of 1885. A description of it has been published in the *Engineer* for June, 1885, and in *Engineering* for July 31st, 1885.

The Tower engine, called by its inventor the "spherical" engine, is based directly on the mechanism of the universal joint. It consists, that is, of a conic crank train of four links, three of them subtending right angles, the fourth subtending a smaller angle, which in this case is always

[1] *Kinematics of Machinery*, pp. 384-399, and plates xxviii.-xxxi.

made 45°. This short link is the fixed one, and takes the form of a closed spherical chamber carrying two shaft bearings. The links *b* and *d* are alike, each of them has for one element a shaft working in a bearing in *a*. For the rest, each is made externally in the shape of a sector of a sphere subtending (at least in its ideal form) an angle of 90°; that is, a quarter of a solid sphere. The link *c*, the cross of the universal joint, becomes a disc piston, pivotted or hinged to the inner edges of the two sectors by two pins, which are, of course, at right angles to each other. Both the shafts *b* and *d* are caused to rotate by the action of the steam; the rotation of one of them (*b*), which is called the main shaft, is kept as uniform as possible by a fly-wheel, and this shaft is the only one to which the resistance is applied. The other shaft (*d*) is called the dummy shaft; its speed of rotation is allowed to vary continually, as the velocity ratio varies, with different positions of the two shafts; but this is no practical drawback because no work is taken off this shaft directly.

Apart from constructional details, the Tower engine is represented by Figs. 299 to 302, which show it in four different positions, one-eighth of a revolution apart, so that Fig. 302 represents a position three-eighths of a revolution in advance of that of Fig. 299. The starting position, Fig. 299, is identical with that of the universal joint in Fig. 288 of the last section, and the identity of the two mechanisms will be clear without any further explanation. The direction of rotation of the shafts is shown by the arrows. The figures make it easy to understand the working of the mechanism as an engine. The disc *c* divides the sphere always into two equal parts, hemispheres. Half of each of these parts is occupied by the sectors, *b* on the one side, *d* on the other. Let us look at the left-hand half

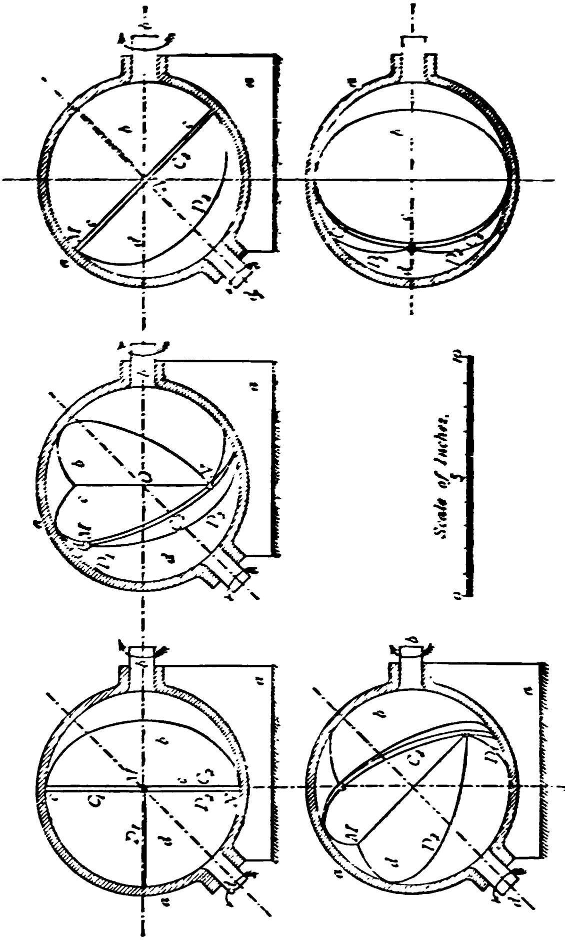


FIG. 299. FIG. 300. FIG. 301. FIG. 302.

Scale of Inches.

alone in the first place, the space in which the *d* sector moves. In Fig. 299 the face D_2 of the sector is close against the lower half, C_2, of the disc and fills the lower half of the hemisphere, the upper half, between C_1 and D_1, being empty. As the position changes to that of Fig. 300, all three links move round, as we know; but, *relatively to each other*, the sector and disc turn about the pivot connecting them, whose axis is, of course, A_{cd}. The sector swings away from the disc, so as to leave a space between C_2 and D_2, and diminish (by an exactly corresponding volume) the space between D_1 and C_1. In Fig. 301 the whole mechanism has made a quarter of a revolution. The pin connecting *c* and *d*, formerly at right angles to the plane of the paper (Fig. 299), now lies in that plane; the disc itself lies in a plane at right angles to the paper. *Relatively to the disc* the sector *d* has turned through 45°, so that it occupies now the centre of its hemisphere, the spaces between C_2 and D_2 and between C_1 and D_1 being equal. The lower half of Fig. 301, (which is a plan projected from the upper half), shows this position clearly. Another eighth of a turn of *b* brings the mechanism to the position of Fig. 302. Here the space between C_2 and D_2 has become nearly as much as that originally between D_1 and C_1, and has come to the upper side, while the originally large space has been continually diminishing. The sum of the two spaces must always be equal to one quarter the whole capacity of the sphere. Another eighth of a turn of *b* would bring the mechanism into a position precisely the same as that of Fig. 299, as far as appearance goes, but with C_1 and D_1 close against each other in the lower part of the casing, and D_2 and C_2 at right angles in the upper part, the whole mechanism having made half a turn. Another half-turn would bring the mechanism back to the position of Fig. 299, the space

between D_2 and C_2 now gradually diminishing and that between D_1 and C_1 gradually increasing. If, when the mechanism is in the position of Fig. 299, any fluid under pressure (the fluid in the Tower engine is of course *steam*) be admitted to the space between C_2 and D_2, and if, at the same time, any portion of such fluid as may be already between C_1 and D_1 be allowed free escape, the pressure of the fluid will force the two surfaces apart, and by so doing cause *d* to turn relatively to *c* and compel the whole mechanism, consequently, to move in the manner we have just noticed. If steam be the working fluid it can be "cut off," as in an ordinary engine, at any point before the half revolution is completed, *i.e.* at any point before the space to which it is admitted reaches its maximum volume of one quarter the volume of the sphere. After cut off, the steam can go on expanding, as the volume of its chamber increases, just exactly as the steam in a cylinder goes on expanding after cut off as the piston moves forward. At half a revolution the steam occupies its maximum volume—the piston has, in effect, reached the end of its stroke. During the next half revolution an opening is provided by which the steam can pass away—exactly as during the return stroke of an ordinary steam piston, until at the end of one whole revolution the whole steam is exhausted, the exhaust passage closed and the admission port opened again for another stroke. On the left side of the disc *c* this whole cycle of processes is gone through *twice* in every revolution, once on each side of the sector *d*. At the same time the same changes go on in the same revolution, on the opposite side of the disc, once on each side of the sector *b*. During each revolution, therefore, the steam is alternately allowed to occupy, and expelled from, *four* spaces, *each* equal in volume to one quarter the sphere. If, therefore, there were no

expansion—that is, if steam were admitted to each space during half a revolution—the volume of steam used per revolution would be exactly equal to the volume of the sphere, a statement which at first sight is apt to appear paradoxical. This condition is not altered very greatly even when the sectors and disc are made in their practical constructive forms, for although the disc *c* has to be of considerable thickness, to allow for steamtight packing round its edge, the faces of the sectors are reduced by an exactly corresponding amount. The space taken up by the joints (or hinges, as Mr. Tower calls them) is, however, permanently unavailable as steam space, and in an engine intended for working at a very high speed this space is not inconsiderable.[1]

In Mr. Tower's engine the link *a* is made the "cylinder" and the link *c* the "piston," an arrangement probably first adopted, although in a much cruder form, by Lariviere and Braithwaite, about 1868.[2]

Mr. Fielding has adopted, in his disc engine, the plan of making the links *b* and *d* the "cylinders," keeping *c* still as the piston, and in this respect, although hardly in any other, (apart from its kinematic identity), his engine resembles that of Taylor and Davies, patented in 1836.[3] The principle of the Fielding engine, apart altogether from any constructive details, is shown in Fig. 303, which is sketched in such a way as to be readily compared with the Tower engine, as well as the mechanisms sketched in the last section. The angle of 45° between *b* and *d*, which is convenient for the Tower engine, would here be inconvenient, and is made only half as great, namely, 22·5°. The link *d*

[1] In the drawings of a spherical engine of 8 inches diameter these hinges are shown as much as 1¾ inches in diameter.

[2] *Kinematics of Machinery*, p. 393, plate xxx.

[3] *Ibid.*

carries two short cylinders, which, from their shape, we may call "ring-cylinders,"[1] D_1 and D_2. The link *b* is precisely similar in shape, the figure shows one of its cylinders only, which lies directly in front of the other in the position shown. The link *c* carries four corresponding "ring-pistons," C_1 and C_2 on the *d* side and two others on the *b* side. These pistons work steam tight in the cylinders with ordinary packing rings, and no other steam packing of any

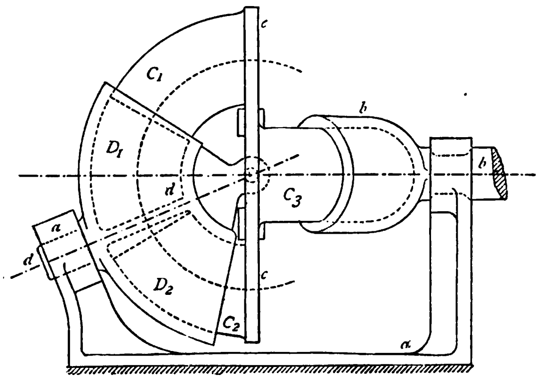


FIG. 303.

complexity is required, a point which seems most advantageous. If the motions of the engine be followed up as we followed those of the Tower engine it will be seen that in each revolution of the engine each piston makes one complete stroke into and out of its cylinder.[2] A very interesting point about this engine is that although the pin

[1] This type of cylinder was first introduced by Mr. Fielding, we believe, in some of Mr. Tweddell's hydraulic riveting machines.

[2] Mr. Fielding has made some of his engines "compound," by making one pair of the cylinders larger than the other, and ingeniously arranging the valves so as to work these as low-pressure cylinders in the usual way.

joints between *b* and *d* and the link *c* are still used, they have become kinematically unnecessary. The pistons C_1 and C_2 are simply an expanded form (§ 52) of the solid of revolution necessary for the turning pair whose axis is A_{cd}, and similarly the cylinders and pistons connecting *c* and *b* form really a turning pair, with axis A_{bc}, connecting these two links. It may perhaps be found possible, presently, to dispense with the pin joints altogether, and make the pairing of piston and cylinder accurate and steady enough to serve instead.

In § 64 we worked out a number of static problems in connection with the universal joint, which of course apply equally to such modifications of the same mechanism as we are now examining. It is worth while, however, to go further than this, and make a complete examination of the engine kinematically, such as we made of some plane mechanisms in Chapter IX. We shall do this with the Tower engine. Once the method of working is thoroughly understood the student will not find it a difficult matter to make a similar analysis of the working of any other engine of the same or similar type. Our object shall be to find the turning effort (or moment) due to the action of the steam upon the main shaft (the link *b*) in a sufficient number of different positions to allow us to represent it by a curve. We shall assume (as with an ordinary engine) that the speed of the main shaft is kept practically constant by a fly-wheel or its equivalent, and shall find the resistances due to the accelerations (positive or negative) of the dummy shaft and sector (link *d*) and the disc or piston (link *c*). We can then put these together to find the real effective moment on the main shaft (as formerly in § 47), from which the necessary size of the fly-wheel can be determined as before, and the whole working of the engine properly analysed.

We know, in the first place, that apart from friction the work in foot-pounds done on the main shaft per revolution must be numerically equal to the whole volume filled with steam per revolution,[1] in cubic feet, multiplied by the mean pressure of the steam in pounds per square foot. This product, multiplied by the number of revolutions per minute and divided by 33,000, gives, of course, the horse-power. In an eight-inch Tower engine the volume of steam used per revolution is about 244 cubic inches, and if we take the pressure as 100 pounds per square inch, continued throughout the whole stroke (*i.e.* without expansion), we get the work done per revolution as

$$\frac{244}{1728} \times 100 \times 144 = 2030 \text{ foot-pounds,}$$

which, at 1000 revolutions per minute, corresponds to over 60 horse-power. We shall use these figures later on as a check on our detailed working. The non-expansive working is only assumed as a simplification. In the Tower engine described in Mr. Heenan's paper the steam was cut off at about half-stroke, and we shall show later on the effect of an early cut-off in the distribution of effort.

The steam pressure which drives the engine acts between the sector and the disc. In any given position of the engine two quadrants have steam pressure in them, as we have seen, one between *c* and *d*, and one between *c* and *b*. In Fig. 299, p. 520, let us assume that steam is just entering between *c* and *d* at the lower side of the engine, and that there is steam also in the space shown to the right of the disc between *c* and *b*. It may save trouble if we call these spaces the *d*-space and the *b*-space respectively. In each case the steam pressure is normal to the

[1] This is ideally, as we have seen, the whole volume of the sphere in the Tower engine.

surface on which it acts, and distributed uniformly over it. By calculation we find that the actual area on which the steam is acting is about 17·7 square inches, and that the radius of the centre of gravity of that area is about 2·2 inches. The total pressure on the piston is therefore 1770 pounds. Instead of taking this pressure as acting at its real radius, it will be convenient to suppose it to act at a radius equal to the radius of the sphere (see p. 511). We may therefore consider that we are dealing with a pressure of $1770 \times \frac{2{\cdot}2}{4{\cdot}0}$ or in all 970 pounds, at a radius of 4 inches. Adopting this simplification we may consider the pressure in the *d*-space all concentrated at the point N, and the pressure in the *b*-space all concentrated at the point M, and amounting in each case to 970 pounds. The point N is a point on the axis A_{bc}, and is therefore a point of b, the main shaft itself, as well as of the disc c. The point M is a point on the axis A_{cd}, and is therefore a point of d, the dummy shaft, as well as of the disc c. It is much more convenient, for geometrical reasons, to consider these points as belonging to b and d respectively, than as belonging to c, and we shall accordingly do so. It will be noticed in such views as Figs. 300 and 302 that the projection of the point N lies always upon a line at right angles to the axis of b, and that of the point M always upon a line at right angles to the axis of d. The real directions of the pressures must always be at right angles respectively to the lines joining N and M to O, the centre of the sphere.

The resultant pressure of the steam in the *b*-space acts always, as can readily be seen, in the plane of the main shaft, and therefore has no moment about it directly. It is, therefore, most convenient to determine first its turning effort upon the link d, and from this find the moment

upon *b* by the use of the (previously calculated) angular velocity ratio of *b* and *d* (see below, Figs. 306-7). The resultant pressure of the steam in the *d*-space acts always, in the same fashion, through the axis of *d*, but it has a moment about the axis of *b* in all positions except those which correspond to the "dead points," that is the beginning and end of the stroke. We shall therefore first determine the action of the steam in the *d* space in turning round *b*; the necessary construction is shown in Figs. 304 and 305.

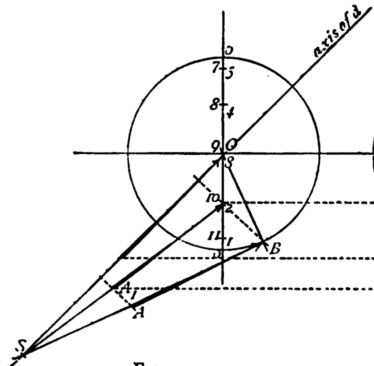


FIG. 304.

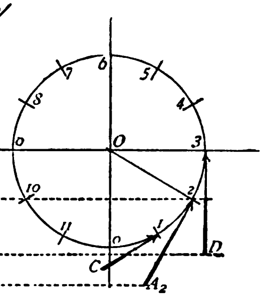


FIG. 305.

Taking (say) twelve equidistant positions of *b* (Fig. 305), we must first determine, by simple projection, the corresponding positions of the point *N*, which are numbered 0 to 11 in Fig. 304. At position 0 we have a "dead point." The pressure at *N* acts in the plane of the paper, and therefore in the plane of the axis of *b*, about which, therefore, it can have no moment. To find the turning effort on *b* at any other position we can have recourse to some such construction as the following, which is worked out for position 2. The position of *N* is known to be that of the point 2 in Fig. 305.

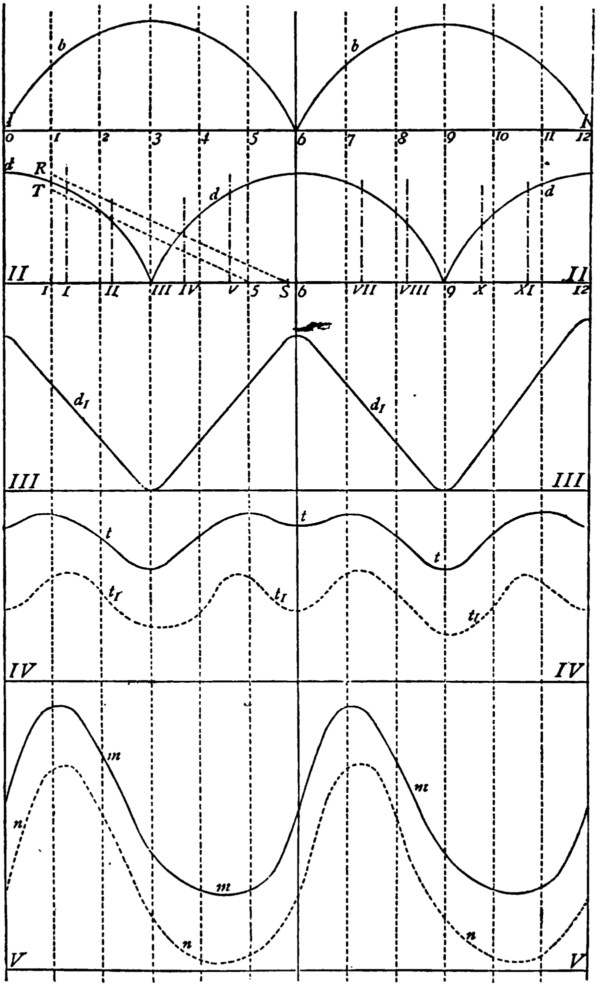
The resultant pressure acting upon it is at right-angles to the line joining 2 to the centre point O, and is at the same time a line in the plane containing the axis of the shaft d and the line $O2$ or ON (compare Fig. 300, p. 520). We can make this plane coincide with the plane of the paper by simply turning it about the axis of d. The corresponding position of the point 2 is found at once by drawing $2B$ (Fig. 304) at right-angles to the axis of d, the point B being on the circle. The pressure will then be represented by a line AB in the plane of the paper and at right-angles to OB, which is of course the projection of $O2$. If the effect of the acceleration of the different masses is *not* to be considered the line AB may be set off to represent the pressure on the disc (either total or per square inch) on any convenient scale whatever. If, however, the accelerative resistances are to be taken into account it is most convenient to work them out first, and use the same scale for piston-pressure as has been used for them. In this case the line AB will represent the equivalent to the total pressure on the piston at its peripheral radius, or, in this instance, 970 pounds. (The scales used here are all shown on the figures.) We have now to turn back the plane, with AB in it, so that the point B comes to 2, and find where A will come. This is easily done; continue BA until it cuts the axis of d (the line about which the plane was turned) in S, and join S to 2. We know then that the projection of A must lie upon the line $S2$, and we know also that it must lie upon a line through A at right-angles to the axis—it is therefore at once found to be A_1. The line $A_1 2$ is therefore the projection in the plane of the paper in Fig. 304 of the whole steam-pressure with which we are dealing. The plane of the paper is a plane *containing* the axis of b. What we wish to find is the component of AB *at right-angles* to that axis; it

therefore only remains to project $A_1 2$ upon a plane at right angles to that axis, and our problem is solved. The plane of Fig. 305 is, as we know, at right-angles to *b*. We have already in that figure the projection of the point 2. We know that the real direction of the pressure lies in a plane at right-angles to the radius $O2$. Its projection in Fig. 305 must therefore be a line at right-angles to that radius, and the position of A_2 can be found at once by projection from A_1. We have, therefore, as the solution of our problem, that the pressure acting between the sector *d* and the piston *c*, when the link *b*, or main shaft, is in the position 2, is equivalent to a force $A_2 2$ acting at a radius $O2$, or (what is the same thing) that the turning moment on *b*, due to the steam-pressure, is $O2 \times A_2 2$. By precisely similar construction $C1$ can be found as the turning effort on *b* for position 1, and if the pressure on the piston is constant (as we have assumed for the present) throughout the stroke, the turning effort at 5 is the same as at 1, and at 4 the same as at 2. For the effort at position 3—as the lines corresponding to BA and $2A_1$ are parallel to each other and to OS,—it only is necessary to set off the pressure magnitude along OS, and project at once to $D3$.

It is hardly necessary to point out that there is no necessity whatever for constructing Fig. 305 separately from Fig. 304, as has been done for clearness' sake. In practice one circle may conveniently be made to serve the purposes of both figures. It may also be noticed that if the pressures at points 4 and 5 differ from those at points 2 and 1, it is still not necessary to make separate constructions for them. The construction shown for position 2, for instance, will serve equally for 4, if instead of AB (Fig. 304) there be set off a distance corresponding to the new pressure, and the corresponding point projected into Fig. 305 instead of A_1.

A diagram of turning effort, similar to the diagram of crank-pin effort formerly drawn (Fig. 155), can now be made for this engine by setting off a base-line representing on any scale the length of the path of the point *N*—*i.e.* equal to the circumference of the circle in Fig. 305, dividing it into twelve equal parts, and setting up at each one, as an ordinate, the corresponding pressure $C1$, $A_2 2$, $D3$, &c. This is done in Fig. 306, *I*, where the similar curve from 6 to 12 is put to complete the revolution.

We have seen that we cannot find directly the turning effort on *b*, due to the steam-pressure between *b* and *c*, because that pressure has no direct tendency to turn *b*, and does so only because it turns *d*, and this rotation cannot occur without the simultaneous rotation of *b*. It is unnecessary to make any further construction to find the turning effort on *d* due to this steam-pressure in the *b* space, for it must be precisely the same as the effort on *b* due to the pressure in the *d* space, which we have just found. Remembering only that 3 and 9 are now the dead points instead of 0 and 6 (as can be seen at once from Figs. 299 and 301), we may therefore set out the turning effort on *d* at once, as in Fig. 306, *II*, using the same ordinates as those we have just found for the line *bb* in the same figure. To combine the two diagrams, however, so as to find the total turning effort on *b*, we cannot simply add the two ordinates together (as we should do if they were diagrams for the two cylinders of an ordinary engine), for the angular velocities of *b* and *d* are, as we know, very different, so that a given effort on *d* may be equivalent to an effort of very different magnitude, although in a corresponding position, on *b*. We must, therefore, find first the angular velocity ratio between *b* and *d* for each of the twelve positions of *b*, by one of the methods of the last section (p. 508), and it is convenient to


b
b
I
I
0
1
2
3
4
5
6
7
8
9
10
11
12
d
R
T
d
d
II
II
I
I
II
III
IV
V
5
S
6
VII
VIII
9
X
XI
12
d_1
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III
III
t
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IV
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n
V
V

FIG. 306.

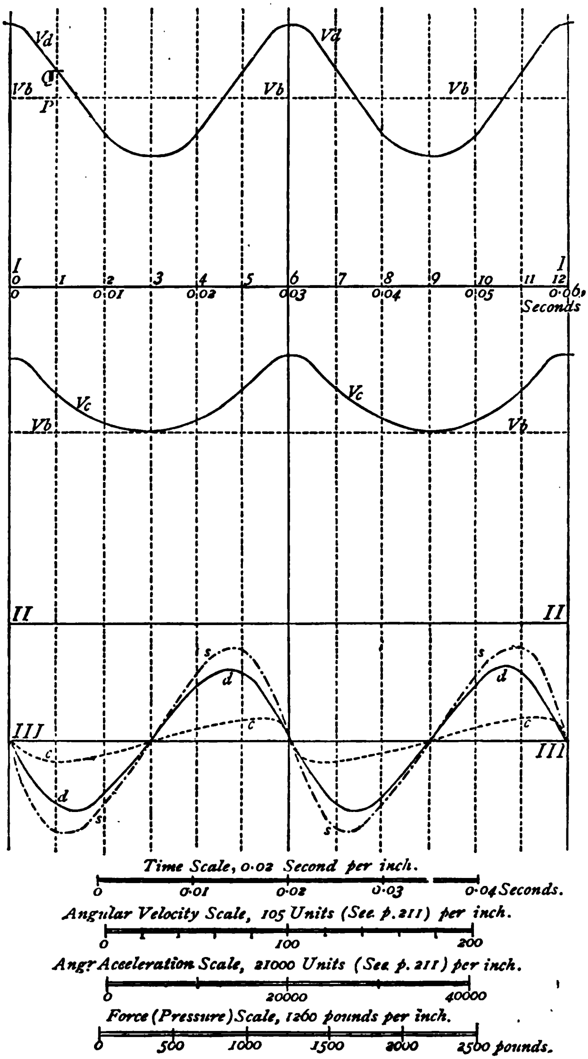

Vd
Vb
P
I
0 1 2 3 4 5 6 7 8 9 10 11 12
0 0·01 0·02 0·03 0·04 0·05 0·06
Seconds
Vc
II
III
s
d
c
Time Scale, 0·02 Second per inch.
0 0·01 0·02 0·03 0·04 Seconds.
Angular Velocity Scale, 105 Units (See p. 211) per inch.
0 100 200
Angr Acceleration Scale, 21000 Units (See p. 211) per inch.
0 20000 40000
Force (Pressure) Scale, 1260 pounds per inch.
0 500 1000 1500 2000 2500 pounds.

FIG. 307.

plot these out in a diagram, as in Fig. 307, *I*. Here any height is taken to represent the (assumed constant) angular velocity of *b*, and the calculated or constructed angular velocities of *d* are set off on the same scale (the lines in the figure are marked v_b and v_d respectively). It will be found convenient for some purposes to represent the angular velocity of *b* by a length equal to that of four of the divisions 01, 12, 23, &c., on the base line. It will be noticed that only four different values of the angular velocity of *d* have really to be found, the rest are all duplicates of these, the velocity at 4, 8, and 10 being the same as at 2, that at 5, 7, and 11 the same as at 1, &c.

It would not be right, however, to take now any ordinate of the *d* curve, and simply multiply it by the ratio $\frac{\angle^r \text{ vel } d}{\angle^r \text{ vel } b}$, and add the ordinate so found to the ordinate of the *b* curve directly above it. For although the ordinate of the *d* curve at 0, for instance, gives the turning effort on *d* which is contemporaneous with the dead point of *b*, the ordinate of the *d* curve at 1 does *not* give the turning effort on *d* contemporaneous with the ordinate of *b* at 1, and the angular velocity ratio between the shafts at 1 in Fig. 307. Contemporaneous points must first be found by the method of the last section (p. 504), and marked on the base line of curve *d*, *I*, *II*, *III*, *IV*, &c. To find the real effort on *b* at position 1, due to pressure transmitted to it from *d* through *c*, it is only necessary to take the ordinate of the *d* curve at *I*, multiply it by the angular velocity ratio at 1, and add the product to the ordinate of the *b* curve at 1. This may perhaps be most rapidly done as follows:—Carry the ordinate at *I* back to 1, as 1*T*. Set off along the base line distances equal to the angular velocities 1*P* and 1*Q* in Fig. 307. (If the dimension for the velocity of *b* has been

chosen as recommended above, it will be unnecessary to measure and set off $1P$, because it will be always equal to the length of four of the equal base line divisions already drawn.) 15 in Fig. 306, *II*, will thus stand for the velocity of b, and $1S$ $(=1Q$ in Fig. 307) for the velocity of d. Drawing SR parallel to $5T$ we obtain $1R$, the required pressure on b in position 1, for by similar triangles $1R = 1T \times \frac{1S}{15}$ = pressure on $d \times \frac{\angle^r \text{ vel } d}{\angle^r \text{ vel } b}$. Carrying out this construction for a sufficient number of points in d, we obtain the ordinates which are plotted out as a new curve $d_1 d_1$ in Fig. 306, *III*. Lastly, adding together the ordinates of b and d_1, we get the curve $t\,t$ (IV), whose ordinates represent the total turning efforts on b at a radius equal (in this case) to four inches, the pressure scale being, of course, still the same as that used originally in Fig. 305.

Going on now to the direct consideration of the effect of acceleration in the engine, we notice at once the general resemblance of the problem to that of § 47. We have here again three moving links, one of which rotates with a velocity assumed to be sensibly uniform. Of the other two one (the link d) moves about a fixed axis[1] with certain large changes of velocity, which we have now completely determined and diagrammed. The remaining link here, the disc c, has motions analogous in certain important points to those of the connecting rod in an ordinary engine, which corresponds to it in being the link which transmits motion to the main shaft of the engine. Both are links which not only undergo varying accelerations, but for which also the accelerations occur about varying

[1] The "reciprocating parts" of an ordinary steam engine are in the same condition, but the axis about which they turn is an infinitely distant one.

axes.[1] The angular velocity of c relatively to b can be found in precisely the same manner as that in which we have found (p. 508) the angular velocity of d relatively to b. We find, namely, the position of the line A_{bc}, which is common to b and c; choosing any convenient point in that line, we find its distances from A_{ac} and A_{ab} respectively; we then say that the angular velocities of the two bodies are inversely as these distances. We thus know, without any construction, that for positions 0 and 6, where A_{ac} coincides with A_{ad}, the link c must have the same angular velocity as the link d, and that for positions 3 and 9, where A_{ac} coincides with A_{ab}, the link c must have the same angular velocity as the link b. The former will be the maximum, the latter the minimum, value of the angular velocity ratio of c to b, and the value of the former we already know (p. 508) to be $\frac{1}{\cos 45^\circ}$, or 1·41, while the value of the latter is unity. The intermediate points in the curve on Fig. 307, *II.* have been found by the following construction, which is analogous to that of Fig. 292, already given. The three virtual axes which we require are A_{ab}, A_{ac} and A_{bc}. We have already seen (Figs. 286 and 290) how to find the position of each. In Fig. 308 they are respectively represented (in position 2 of b) by the lines OP, OQ and OR. These three lines are in one plane (p. 490), and the points P, Q, and R lie all in one circle. The real relative position of the axes is therefore obtained at once by simply turning down the plane about OP until it coincides with the plane of the paper, so that Q comes to Q_1 and R to

[1] The motion of the disc has its most exact representative in plane mechanisms by the motion of the link c in the chain of Fig. 32, supposing it converted into a mechanism of which a is the fixed link, a mechanism on which innumerable rotary engines have been based.

R_1. We then have at once $\frac{\angle^r \text{ vel } c}{\angle^r \text{ vel } b} = \frac{OR_1}{SR_1} = \frac{1}{\cos POQ}$. It will be noted that, just as in the upper curve in Fig. 306, the twelve points on the curve of angular velocity of c require only the calculations of *four* different ordinates, and of these we know one (for positions 0 and 6) to be equal to an ordinate of the former curve, and another (for positions 3 and 9) to be equal to unity.[1]

The two curves of velocity which we have now drawn may be considered as curves drawn on a *time base* (p. 194), for the equal abscissæ 01, 12, 23, &c., correspond to equal motions

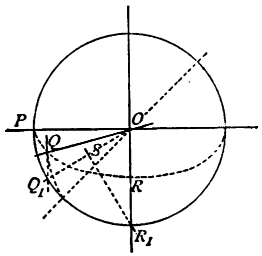


FIG. 308.

of a body (b) whose velocity is uniform, and therefore to equal intervals of time. If we take the engine as making 1000 revolutions per minute, one revolution occupies 0·06 second, and each of the twelve divisions of the base line corresponds, therefore, to 0.005 second. Using the method given in § 28, p. 194, for finding the acceleration from a given velocity curve on a time base, we can now draw the curves of Fig. 307, *III.*, of which *dd* represents the acceleration of

[1] For positions 0 and 6, the angle $POQ = \theta$ (the angle between the shafts) $= 45°$, and $\cos POQ = 0·707$; for position 3 and 9 the angle $POQ = 0$, and $\cos POQ = 1$.

the link d, cc that of the link c, and ss the sum of the two accelerations. Ordinates above the axis are positive, that is, they correspond to an *increasing* velocity, ordinates below the axis are negative, corresponding to a *decreasing* velocity.

We know that acceleration curves can be read off at once as curves of force or (in this case) pressure (p. 339). We have only, therefore, to determine the scale on which our curves may be so read in order to compound them with the pressure curves of Fig. 306. We saw in § 31 that

$$f\,r\,t = v_a\,I,$$

where f was the force which, applied at radius r for a time t could produce an angular velocity v_a in a body whose moment of inertia about its virtual axis (from which also r was measured) was I. The angular acceleration $a = \frac{v_a}{t}$, so that $f\,r = a\,I$, or $f = \frac{a\,I}{r}$.

In the diagrams as drawn in Figs. 306 and 307 the angular velocity of b is represented by a height of one inch. The assumed velocity of 1000 revolutions per minute $= 1000 \times \frac{2\pi}{60} = 105$ angular units (per second), so that the scale of angular velocity is 105 units per inch. Four divisions on the time scale (*i.e.* the distance 04, &c.) are made equal to one inch, so that the time scale is 0·02 second per inch. Each time interval being $\frac{1}{200}$ second, the acceleration scale[1] is $(105 \times 200) = 21{,}000$ units per inch. The value of I for the sector d (the units being feet and pounds) is about 0·02, and r, the radius at which we have assumed the pressure to act, is $\frac{1}{3}$ foot. We have, therefore, $f = a \times 3 \times \cdot02 = \cdot06\,a$, so that the force scale is $21{,}000 \times \cdot06 = 1260$ pounds per

[1] See p. 199.

inch.[1] It has been already (p. 529) pointed out that it is convenient to calculate this scale before setting off the piston pressures (Fig. 306), and use it for them. If this has not been done the ordinates *ss* of the total acceleration curve must be reduced to the scale used for Fig. 306 before being further used.

The scale of the accelerations of *c* differs from that of *d* because of the different value of *I*. Approximately the value of the moment of inertia of *c* is half as great as that of *d*, or 0·01, so that the acceleration ordinates derived direct from the *c* curve (Fig. 307, *II*) have to be halved before being set off in Fig. 307, *III*. This has been done before plotting them in the curves shown. It has to be noticed that the value of *I* for the disc *c* is *not* a constant quantity, for the virtual axis (about which *I* is to be measured) does not occupy a constant position in the body. This case is altogether analogous to that of the connecting rod discussed in § 49. The error caused by assuming the value of *I* to be constant is too small to be of any importance to us, and we have therefore neglected it. Practically a very reasonable approximation to the result is obtained by making the same assumption about the link *c* that is commonly made about the connecting rod of a steam-engine, namely, that half its mass shares the motion of the main shaft, and has therefore no acceleration, and that the other half may be taken as part of the mass of the link *d*, and as sharing its accelerations. If this approximation were used in the present case it would give a total acceleration about $\frac{12}{18}$ as great as that found by our more exact method. The saving of trouble by the

[1] It will be noticed that the pressures we are here dealing with are *total* pressures at an assumed radius, *not* pressures reduced to unit area of piston and mean radius. The reduction can easily be done (see p. 527) if required

omission of the construction of Fig. 308, and the curve Fig. 307, *II*, is of course considerable.

Going back now to Fig. 306, *V*, we have in the curve *mm* the sum of the ordinates of the curve *tt* above it, and of the curve *ss* in Fig 307, *III*, (the two curves being supposed to be drawn on the same pressure scale). The extraordinary result of the accelerative resistances we see at once. Whereas, without them the driving effort was fairly uniform, the ratio of maximum to minimum being about 1·5, when we take into account the accelerations at 1000 revolutions per minute this ratio is increased to about 3·4. The dotted curve t_1t_1 shows the variation of nominal driving effort, if the steam were cut off at about $\frac{1}{4}$ stroke, and *nn* the real effort under the same conditions at 1000 revolutions per minute, the ratio just mentioned being increased from about two to over twenty. Large as these alterations are, it will be seen, from the table on p. 352, how much larger changes would be produced in any engine of the ordinary type if it were run at anything like the same number of revolutions per minute. The advantage here is no doubt mainly due to the fact that the links which oscillate or swing relatively to each other are both in continued rotation relatively to the fixed link. Thus, although the reciprocating motion is not really done away with, one of its most serious drawbacks is obviated—the reciprocating links do not come to rest, relatively to the fixed link, twice in every revolution, as do the piston and rod, &c., of an ordinary engine.

As a check on the working and diagramming the *mean* effective turning effort ought to be measured from the curve in Fig. 306, *V*. It should be, and in this case is, equal to the known mean steam pressure at 4 inches radius, or here 970 pounds, which corresponds to 2030 ft.-pounds per stroke.

The working out of a Fielding engine may proceed in precisely the same way as that which has been employed for the Tower engine. It is not necessary here to say anything further than that this engine has the advantage, from the point of view of steady running, that the angle between the shafts is much less than in the spherical engine, $22^\circ{\cdot}5$ instead of 45° being used. This makes the ratio of maximum to minimum angular velocity of the dummy shaft $\left(\frac{1}{\cos^2\theta}\right)$, only $1{\cdot}17$ instead of 2. Supposing the masses of the rotating parts to be the same in both engines, the resistances due to acceleration at any given speed will be about $\frac{1}{5}$ of the amount of those just diagrammed. But at the same time an engine of the Fielding type has a net volume for steam less than that of a Tower engine (both being of the same external diameter) in somewhere about the same ratio. The enormous effective volume of the latter engine depends essentially, as we have seen, on the use of a large angle between the shafts, and this unavoidably entails irregularities of driving effort due to the great accelerative resistances, which have to be rendered as unimportant as possible by the use of sufficiently heavy rotating masses (flywheel or its substitute) upon the driving shaft.

§ 66.—BEVEL GEARING.

If we treat the spur wheel chains of Chapter VI. as we have just treated the linkwork mechanisms of the earlier chapters—if, namely, we transform the plane into spheric motion, by bringing the point of intersection of the axes to a finite distance, we obtain the type of wheel gearing known

as **bevel gearing.**[1] It is possible to reproduce, in this form, *all* the spur wheel chains, simple, compound, annular, or epicyclic. Practically, however, very little use is made of any of these changed mechanisms, except the simplest of all, which is shown in Fig. 309, and which is directly derived from the spur wheel train of Fig. 58, p. 117, by inclining its shafts at an angle (in this case 45°) to each other. We have just the same characteristics here as formerly with the spur wheel train; the one shaft drives the other with a constant velocity ratio, but in the opposite sense to that in which it is itself rotating. The portions of the surfaces of the cylindric axodes, which we saw to be

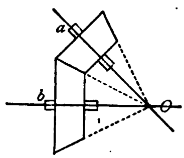


FIG. 309.

formerly the pitch surfaces, are now replaced by portions of the conic axodes. The motion of the toothed bevel wheels corresponds to that of the rolling of the conic pitch surfaces, just as formerly the motion of the spur wheels corresponded to the rolling of the cylindric pitch surfaces. We form teeth on the former for exactly the same reason as we did on the latter; and the actual transmission of motion is accompanied by the sliding on one another of these teeth, exactly as we saw formerly.

Under these circumstances it is not necessary for us to say more than a very few words about this form of non-

[1] If a pair of bevel wheels are of the same size they are often called *mitre wheels*.

ne motion. It will be noticed at once that just as the ks in the conic chains had no dimensions which could operly be called their *lengths* (p. 493), so here the wheels ive no one special diameter.[1] Wheels of the most various iameters may transmit the same velocity ratio between the same two shafts. Thus let a and b (Fig. 310) be the axes of two shafts intersecting in O, between which it is desired to transmit a known velocity ratio. Let this ratio be such that $\frac{\text{ang. vel. } b}{\text{ang. vel. } a} = \frac{OA}{OB}$. Draw about the point O any circles with radii OA and OB; through A draw a parallel to the axis

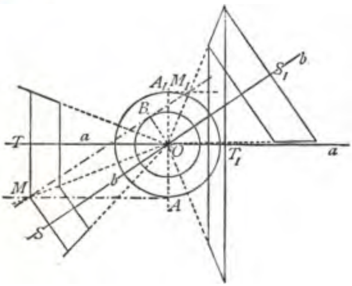


FIG. 310.

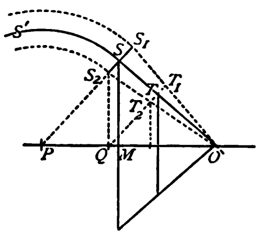


FIG. 311.

a, and through B to the axis b. Call the point where these parallels intersect M, and join MO. Then MO is the line of contact of the two pitch cones, which can, therefore, be at once drawn. Any pair of frustra of these may be used for the wheels, for the ratio $\frac{MT}{MS}$ between their radii must always be the same, and must always be equal to $\frac{OA}{OB}$.

The finding of the shape of the teeth for bevel wheels

[1] Conventionally the largest pitch diameter of a bevel wheel is spoken of as the diameter of the wheel.

involves no difficulty. Let O (Fig. 311) be the vertex for such a wheel, MS its radius, and S_1S_2 the required depth of tooth.[1] We may treat the spheric surface S_2SS_1 as if it were itself a part of a cone with vertex at P ($OSP = 90°$) complementary to the pitch cone. This cone can be developed by drawing circles through S_2, S, and S_1 with P for a centre. If the circle SS' be now used as a pitch circle, and teeth drawn on it (see § 18) with the right depth in the usual way, the profiles of these teeth will be the profiles required. The corresponding profiles for the inner sides of the teeth can be found by developing the cone TQ in exactly the same fashion. It is hardly necessary to point out that all lines along the teeth,

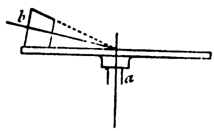


FIG. 312.

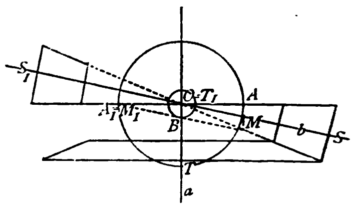


FIG. 313.

such as S_1T_1, S_2T_2, &c., must pass through O as well as the line ST on the pitch surface. Annular bevel wheels, although they are kinematically quite correct, are rarely, if ever, used. They come out at once from the construction of Fig. 310, if only the point A_1, at the opposite end of the diameter, be used instead of A. In the event of the angle M_1OS_1 being equal to the angle A_1OS_1, the annular wheel becomes a disc or flat wheel, as shown in Fig. 312, which may often be a quite convenient arrangement of gearing. Fig. 313, which corresponds to Fig. 310, shows the double

[1] Determined mainly by considerations of strength, &c., such as will be found discussed in Chap. ix. of Professor Unwin's *Elements of Machine Design*, &c.

construction in this case. It will be noticed that, for any given sense of rotation of *b*, the shaft *a* will be driven by the flat wheel in the opposite sense to that in which it would be driven by the cone wheel, which is, of course, the essential characteristic of an annular train.

It is comparatively easy to make correctly shaped *patterns* for the teeth of bevel wheels, and the shape of the bevel tooth, when the wheel is cast, will be fairly near the shape of the pattern. But if it is required that the profiles of the teeth should be really accurate, they must be, as with spur gearing, *machined*, and this operation is one presenting some practical difficulties. The most recently devised machine for this purpose, a very ingenious one, is probably that of Mr. Bilgram, described in *Engineering*, vol. xl. p. 21, the principle of which will repay examination.

§ 67.—THE BALL AND SOCKET JOINT.

THE familiar combination known as the **Ball and Socket Joint** is not, as might be at first sight supposed, a pair of elements. It does not *constrain* any relative motion between the bodies which it connects; it only permits the one to have spheric motion, in any direction, relatively to the other. It cannot, therefore, be used as the sole connection between two bodies in a mechanism or machine unless the relative motions of those bodies be completely constrained by what we have called *chain closure* (p. 410), or its equivalent. In that case it forms, essentially, an example of *reduction* (p. 403). The links connected by the ball and socket joint could not be directly connected by any one lower pair, but might be connected with the use of lower pairs only if one or more links were inserted between them,

exactly as in the cases examined in § 53. The ball and socket joint, however, could not be replaced by *plane* links; its motion is spheric, and the links which it virtually replaces would have to form some part of a spheric combination. The ball joint has, of course, *surface* contact, but as it is not a pair of elements this is no contradiction to the statement on page 57, that none but lower pairs of elements had surface contact.

Fig. 314 shows a mechanism which has occasionally found application, and which belongs to a class which will be mentioned in § 70. It is a *simple* chain, each link having

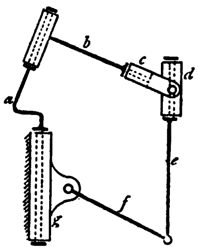


Fig. 314.

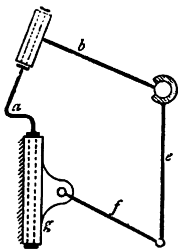


Fig. 315.

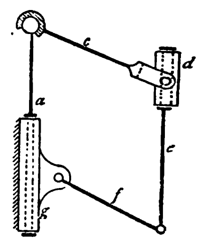


Fig. 316.

only two elements. It contains seven links, each paired to its neighbour by a turning pair. The axes of three pairs, namely, *bc*, *cd*, and *de*, pass through one point, an arrangement which so constrains the relative motion of *b* and *e* that the axes of these two links always intersect, and always intersect *in the same point*. The motion of *b* relatively to *e* is therefore a spheric motion about that point. If we wish to dispense with the links *c* and *d* we may proceed as in § 53, by forming on *b* a suitable element, and finding its envelope on *e*. If we choose for the element a sphere whose centre is at the join of the axes of *b* and *e*, its

envelope on the latter link will obviously be simply a corresponding hollow sphere, and the mechanism, so reduced, will take the form of Fig. 315. The motions of the remaining links are here sufficient to constrain the relative motions of *b* and *e*, and the ball joint becomes in this very special case available for use as a higher pair. The motion of *c* relatively to *a* is also such that a certain line on *c* (namely, the axis of the pair *bc*) always passes through the same point on *a*. We might, therefore, omit *b*, and pair *c* to *a* (as in Fig. 316) by a ball joint. By doing this, however, the chain becomes unconstrained, for the link *a* can be rotated without transmitting any motion to the rest of the chain, the ball joint being incapable of transmitting rotation about its own centre. The investigation of the conditions under which a ball joint can be used in a reduced chain without destroying its constrainment does not present any great difficulties, but the case is one which occurs so seldom that we shall not here enter into it.

§ 68.—HYPERBOLOIDAL OR SKEW GEARING.

THERE remains yet to be noticed a class of mechanisms having non-plane motions, but coming under none of the categories hitherto examined in this chapter. These mechanisms may contain only turning pairs (as for example the one illustrated in the last section), or they may contain also screws or cams or higher pairing of any kind. Their characteristic is that some or all of their links have, relatively to some of the other links, a general screw motion, for which reason we may give them the generic name of **general screw mechanisms.** By general screw motion is meant a twist which bears the same relation to simple screw motion

that rotation about an instantaneous axis bears to rotation about a permanent axis. A body having such a motion is at each instant twisting relatively to (say) the fixed link. But both pitch and axis of twist may, and often do, vary from instant to instant. There is here neither virtual centre nor virtual axis, neither centrode nor axode, cylindric or conic. The motion cannot be represented as a rotation, or by a *rolling* of two surfaces of any kind. For any two bodies having general screw motion relatively to each other, it is always possible (but sometimes extremely difficult) to find a line that is common to both the bodies for the instant, and about and upon which each is simultaneously *turning* and *sliding*, at the instant, relatively to the other. Such a line may be called the axis of virtual twist, or simply the **twist-axis**, of the two bodies. As an axis, it may be said to be a line common to the two bodies, but it is not a common line in the same sense as the virtual axis of rotation, for as a line in one body it slides along its own direction relatively to the other body. The complete series of twist-axes for the relative motions of two bodies, that is, the loci of these axes, form a pair of ruled surfaces (**twist-axodes**) which may be properly looked upon as the general case of the simpler axodes of plane and spheric motion. As the bodies move, successive lines on these surfaces come into coincidence, and the motion of the one body relatively to the other is always a twisting about the coincident line, exactly as in plane motion there is always a rotation about the coincident line of the two axodes.

A detailed study of these mechanisms—a subject in which comparatively very little work has yet been done—would take up an amount of space altogether out of proportion to their importance, for their applications in practical machinery are comparatively few, and their complexities, from anything

like a general point of view, are very great. We shall here not attempt any such complete examination (perhaps we may have some future opportunity of dealing with the subject), but shall merely mention a few of the principal examples which occur in actual work. The simplest of these cases occurs where the twist is the same for each twist-axis, and of these cases the simplest again is no doubt the one

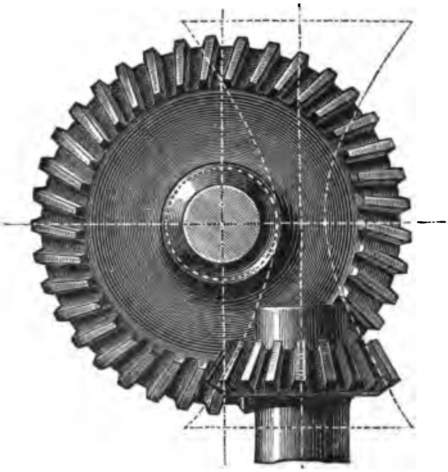


FIG. 317.

(analogous to spur and bevel wheel gearing) where the twist-axodes are used directly, altered only by being toothed, for the transmission of rotation between shafts whose axes cross, but do not meet, each other. Gearing of this kind (an example of which is shown in Fig. 317) is known as **skew wheel gearing**, the wheels being often called, from their quasi-conical form, **skew bevel wheels**. The twist-axodes, frustra of which correspond to the pitch surfaces of the skew

bevel wheels, are hyperboloids of revolution whose axes are the axes of the shafts and which have always one coincident line or generator in a position corresponding to the coincident line of the pitch surfaces of spur wheels. The position of this line must be such that the distances of every point in it from the two axes must be inversely proportional to the required angular velocities of the wheels, and it can be found in the following manner. Let SA and SB (Fig. 318) be projections of two crossed axes a and b, both parallel to the plane of the paper, so that the common normal to the two axes passes through S and is normal to the plane of the paper.

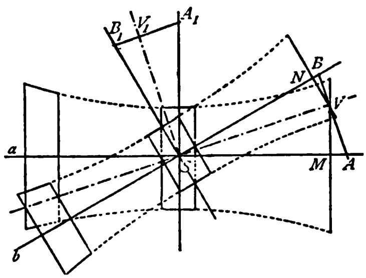


FIG. 318.

Let the angular velocity ratio of b to a be given, and let it be required to find the generator for the (hyperboloidal) pitch surfaces of skew bevel wheels which will transmit this ratio. The angle ASB must first be divided by SV so that $\frac{VM}{VN} = \frac{\text{ang. vel. } b}{\text{ang. vel. } a}$, which can be done exactly as described for bevel wheels in § 66, Fig. 310. Then SV is the projection of the required generator, which lies in a plane parallel to that of the paper as drawn. Drawing through V a line at right angles to SV, we have at once $\frac{VA}{VB}$ as the constant

ratio of the distance of every point in the generator from the axes *a* and *b*, and therefore the required ratio between the radii (or diameters) of *a* and *b*, which are no longer proportional themselves to the velocity ratio.

It requires now to be proved that skew wheels with this diametral ratio will transmit the required velocity ratio. Draw SA_1, SV_1, and SB_1 at right angles to SA, SV, and SB respectively, and consider the contact between the pitch surfaces upon the normal to the shafts, *i.e.* upon the line through S and normal to the plane of the paper. The direction of the line of contact is SV, and both wheels must have the same velocity normal to that direction.[1] Drawing A_1B_1 parallel to SV, we obtain at once SA_1 and SB_1 as the peripheral velocities of *a* and *b* respectively, on any scale on which SV_1 represents their common velocities in its own direction. The angular velocities of the two wheels must be directly as their peripheral velocities and inversely as their radii, or

$$\frac{v_b}{v_a} = \frac{SB_1}{SA_1} \cdot \frac{r_a}{r_b} = \frac{SB}{SA} \cdot \frac{r_a}{r_b} = \frac{VA}{VB} \cdot \frac{SB}{SA}.$$

$$\text{But } \frac{VA}{VM} = \frac{SA}{SV}, \text{ and } \frac{VB}{VN} = \frac{SB}{SV} \text{ so that}$$

$$\frac{VA}{VB} = \frac{VM.\ SA}{VN.\ SB},$$

$$\text{and } \frac{v_b}{v_a} = \frac{VM.\ SA}{VN.\ SB} \cdot \frac{SB}{SA} = \frac{VM}{VN},$$

which is the required ratio with which we started.

Any pair of corresponding sections of the hyperboloids may be used for pitch surfaces, two pairs being shown in

[1] Or put otherwise, instead of the two wheels having the same peripheral velocity, as with spur gearing, they have *different* peripheral velocities, but these different velocities must have *equal components along the direction* SV_1.

the figure. The directions of the flanks of the teeth on the pitch surfaces must correspond to the directions of the generator. Frequently frustra of tangential cones are employed in this gearing instead of frustra of the hyperboloids. In that case the shape of the tooth profiles is obtained by designing them in the way described for bevel wheels on p. 544.[1] If sections from the *throats* of the hyperboloids be chosen, the teeth may be made of uniform section right across, like those of spur wheels, but, of course, skewed at the proper angle. The ratio between the numbers of teeth in the wheels must be proportional (inversely) to their intended velocity ratio, and *not* proportional directly to their diameters. The pitch of the teeth on the two wheels, measured circumferentially, is of course different. If SV_1 be the normal pitch (*i.e.* the pitch measured at right angles to the face of the tooth), which is *the same in both wheels*, then SB_1 must be the circumferential pitch of b and SA_1 of a.

If the perpendicular distance between the shafts be t, then the diameters of the two pitch surfaces at the throats, or smallest parts, are respectively,

$$t.\frac{VA}{AB} \text{ for } a, \text{ and } t.\frac{VB}{AB} \text{ for } b.$$

At any other places the diameters can be found from the data in the figure by the ordinary projective constructions.

If the distance SV_1 represent on any scale the common peripheral velocity of the two wheels normal to the direction of the twist axis, then the distances V_1B_1 and V_1A_1 represent on the same scale their velocities of sliding along that axis. The velocity with which each one slides relatively to the other is therefore $B_1V_1 + V_1A_1$, or B_1A_1. This corresponds,

[1] A more exact approximation will be found in *Der Constructeur*, third edition, p. 452, or fourth edition, p. 553.

of course, to the axial component of the twisting motion of the axodes. In this, the simplest case of general screw motion which we have in machinery, the magnitude of the twist (which may be expressed conveniently enough as the ratio $\frac{SV}{BA}$) is the same for each twist axis, as we have already noticed.

We may consider that spur wheels are the special case of skew wheels where SB coincides in direction with SA, and where, therefore, $B_1A_1 = O$. Looking at matters in this way, we see that the teeth of skew wheels must have the same rubbing action in planes normal to their lines of contact (*i.e.* normal to SV) as ordinary spur wheels, and *in addition* a rubbing action in the direction of those lines. As all such action involves the expenditure of work in overcoming the frictional resistances which the surfaces offer to sliding one on the other, the frictional losses in skew wheel gearing are necessarily considerably greater than in spur gearing. More will be said about this matter in the next chapter.

A skew-wheel train may, of course, contain an annular wheel, or it may be made epicyclic by fixing one of the wheels instead of the frame. Practically an annular skew wheel would be so troublesome to make that we are not likely to see one, especially as it is always possible (as we have seen with bevel wheels) to alter the sense of rotation transmitted without the use of an annular wheel. There is no particular difficulty about making an epicyclic skew train, but no occasion seems yet to have occurred for using one.

§ 69.—SCREW WHEELS.

IF we desire to drive two crossed shafts, one from the other, by a pair of wheels whose teeth can always touch each other *along a straight line*, we have no alternative but to use the skew wheels described in the last section. We must, moreover, use for their pitch surfaces hyperboloids generated by the revolution of one particular line (fixed as we fixed SV in Fig. 318) for each particular angular position of the shafts

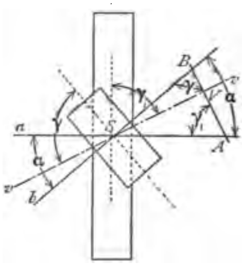


FIG. 319.

and velocity ratio to be transmitted. If, however, we are content to transmit motion through teeth which touch each other on *one point only*[1] at each instant, we have a much larger choice of possibilities. Thus if in Fig. 319, SA and SB are again projections of the axes of crossed shafts (drawn in the same way as in Fig. 318, § 68), we may take any line SV between SA and SB, or *in coincidence with either of them*, as the common tangent at S to screw lines drawn on cylindrical surfaces which have a and b as their axes, and

[1] In one point (for each pair of teeth), speaking kinematically only. Physically, of course, the point becomes a small but undefined area.

which touch at some point on the normal through *S*. If on such cylinders, or slices of them, we build up helical teeth corresponding to the assumed tangent, we shall have what are called **screw wheels**, each wheel being, in effect, a portion of a many-threaded screw.

If the tangent *SV* be taken in the same position as that of the coincident line in skew wheels, the pitch diameters of the screw wheels, for a given velocity ratio, will be the same as those of the throats of the corresponding hyperboloids. In this case the screw wheels will differ very little in appearance from the skew wheels. In the former, however, the faces of the teeth will be helical instead of straight, and in actual working, contact between any pair of teeth will begin on one side, pass through the point *S*, and end on the other side, instead of taking place simultaneously all across the teeth.

For any given velocity ratio, whether the common tangent in mid-position be taken as mentioned in the last paragraph or not, the hyperboloids having a coincident generator, as found in § 68, will remain the twist-axodes for the motion transmitted by the wheels, no matter what the diameters of the latter may be. These diameters are easy to find in any case. Let *V* be any point upon the (arbitrarily chosen) common tangent *SV*, and *BA* a line through *V* normal to *SV*. Then we have already proved (p. 551) that (using the same symbols as before)

$$\frac{r_a}{r_b} = \frac{v_b}{v_a} \cdot \frac{SA}{SB} = \frac{v_b}{v_a} \cdot \frac{\sin\gamma}{\sin\gamma_1}.$$

If, then, *SV* coincide with *SB* (Fig. 320), $\gamma = 90°$, and $\gamma_1 = 90° - \alpha$, and the ratio between the diameters of the wheels is greater than the velocity ratio, a condition in itself disadvantageous. The teeth on *b* are here parallel to its axis, as in a spur wheel. If *SV* bisect the angle *ASB* (Fig. 321)

$\gamma = \gamma_1$, and the diametral ratio corresponds, as with spur wheels, to the velocity ratio. This is the condition of minimum friction. If SV coincide with SA (Fig. 322) the teeth on the a wheel are parallel to its axis, the angle $\gamma = 90^\circ - \alpha$ and $\gamma_1 = 90^\circ$, and the diametral ratio is *less* than the velocity ratio. Intermediate positions have, of course, corresponding characteristics. If V be nearer B than A, the diametral ratio is greater than the velocity ratio, and *vice versâ.*

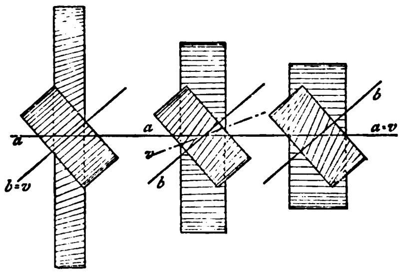


FIG. 320. FIG. 321. FIG. 322.

In the most common case of screw wheels occurring in practice, the angle α, between the shafts, is a right angle, and the velocity ratio transmitted is large. In this case the combination becomes the worm and worm-wheel (Fig. 323), which we have already looked at in § 62 from another point of view. If we were here to make $\gamma = 90^\circ$, the pitch of the helix on a would become $= 0$, and that on $b = \infty$. If we were to make $\gamma_1 = 90^\circ$, the pitch on a would become $= \infty$ and on $b = 0$. In neither case, therefore, would the mechanism work. In practice γ is made some small angle, and (as $\gamma + \gamma_1 = 90^\circ$) γ_1 is a much larger one. A very large velocity ratio can therefore be transmitted by a pair of screw wheels (as the worm and wheel really are) of much

smaller diametral ratio. Thus, for example, if $\gamma = 10°$, γ_1 must $= 80°$, the value of $\frac{\sin\gamma}{\sin\gamma_1} = 0{\cdot}175$. In such a case any velocity ratio r can be transmitted by a worm and wheel whose diameters are in the ratio of $(0{\cdot}175\ r)$ to each other. This is, of course, a great practical convenience. If a pair of skew wheels were used under similar conditions, with a value of r of 50, their diametral ratio would have to be 50, instead of $0{\cdot}175 \times 50$, or $8{\cdot}75$.

In both cases equally the number of teeth in the wheels must be proportional to the velocity-ratio transmitted, but with screw wheels the number of teeth means *the number of threads in the screw*, for that is the real number of teeth that would be shown by any section of the wheel normal to its axis. A single-threaded screw, such as is often used for a worm, if cut by a plane at right angles to its axis, would show only *one* tooth and one space—it is in reality a one-toothed wheel. A double-threaded screw, similarly, is equivalent to a wheel of two teeth, and so on.

The *pitch* of screw wheels has to be determined in the same way as that of skew wheels. The pitch *of the teeth*, measured at right angles to the common tangent, must be the same in both wheels, or in wheel and worm, but this quantity must not be confused with the pitch *of the screw lines*. The real pitch of the teeth in screw wheels is the distance represented by the spaces between the lines in Figs. 320 and 322 of this section. This is determined from the normal pitch exactly as on p. 552, and we do not concern ourselves at all with the relation between this (circumferential) quantity and the (axial) pitch of the helices on which the threads are formed, which is in these cases always a very much *larger* distance. In the case of the worm of Fig. 323, the case is, however, reversed. Taken as a single-threaded

screw, or one-toothed screw wheel, the pitch of its tooth is equal to its circumference, while the pitch of the helix, or distance from one convolution to the next, measured axially, is a much *smaller* quantity. It is the former, or tooth-pitch, which is given by the calculation on p. 552. In this particular case, however, it is the helical pitch which is the visible thing, and the real nature of the worm as a screw wheel is sometimes obscured by the confusion between the two pitches. In this particular case the circumferential

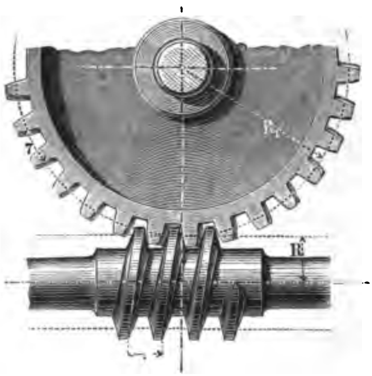


FIG. 323.

pitch of the worm wheel is equal to the axial pitch of the worm helix if it be single threaded, to half that pitch if it be double threaded, and so on.

We have already mentioned (p. 487) the Sellers worm gearing, in which the shafts are set at an angle less than 90° by an amount equal to γ, so that $\gamma_1 = \alpha$, SV coinciding with SA. In this case the teeth of the wheel, like those of the wheel in Fig. 322, lie parallel to its axis; the wheel in fact simply becomes, or may become, a spur wheel, and its

construction is correspondingly simplified. The form of the teeth of worm wheels has already been mentioned in § 62.

We have stated on p. 554 that the tangent line of screw wheels may be taken anywhere between *SB* and *SA*. Kinematically it may be taken outside these limits also, but in this case the friction due to the sliding of the teeth (see p. 553) becomes excessively great, without counterbalancing advantage of any other kind.

Longitudinal sliding of the teeth on one another produces additional friction in screw wheels (as compared with spur or bevel wheels) exactly as in skew wheels, but not to the

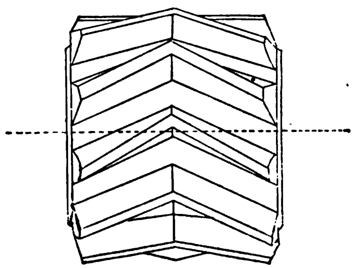


FIG. 324.

same extent, the area of surface in contact being much smaller (see p. 554). In both cases also the obliquity of the pressure causes end thrust in the journals of one or both of the shafts. This is often more serious in screw than in skew gearing, because of the greater obliquity. If the shafts are *parallel*, this difficulty can be got over by the use of the double-helical wheels of Fig. 324, which were mentioned in § 19 (p. 131). These wheels, although they are used with parallel shafts, are real screw wheels; the contact of the teeth is a point contact only, and not a line contact, and there is always contact in at least one pair of points

along the pitch line. They present certain practical difficulties in manufacture, but these have been long ago overcome, and very large numbers of them are used on the Continent, and also by some English makers. The small surface of contact makes them work very "sweetly" if the teeth are reasonably well formed. It will be noticed that their relative motion is represented simply by the rolling of cylindrical pitch surfaces, or axodes, as with spur wheels, and not by the twisting together of hyperboloidal surfaces, as in the case of screw wheels with crossed axes or skew bevel wheels. The cylindrical pitch surfaces here show themselves at once as being a special case of the hyperboloidal surfaces.

§ 70.—GENERAL SCREW MECHANISMS.

THERE remain to be mentioned **general screw mechanisms** (see p. 547) of a much more complex kind than the hyperboloidal or screw wheels of the last two sections. Of such mechanisms two have been already illustrated in Figs. 314 and 315 in § 67, a third is shown in Fig. 325. Of these a modification of Fig. 314 has found actual, if not very practical, application in machines. The other two have not, so far as we know, had any practical applications. A general investigation into their conditions of constrainment, or determination of the twist-axodes of the different links, does not appear as yet to have been made. Such a determination is not very difficult in the case of Figs. 314 and 315, where also the mechanism can be drawn in any position, without any difficulty, by the ordinary constructions of orthographic projection. With Fig. 324, however, these constructions alone do not enable the motions of the mechanism to be

drawn. The case appears to be analogous, among non-plane mechanisms, to the case of the third order in plane mechanisms which was discussed in § 59 (p. 458). The extremely limited practical importance of these mechanisms makes it unsuitable that anything in the nature of a general discussion of their properties should be attempted here. It is much to be hoped that some competent geometer will presently take them in hand, and classify and analyse them.

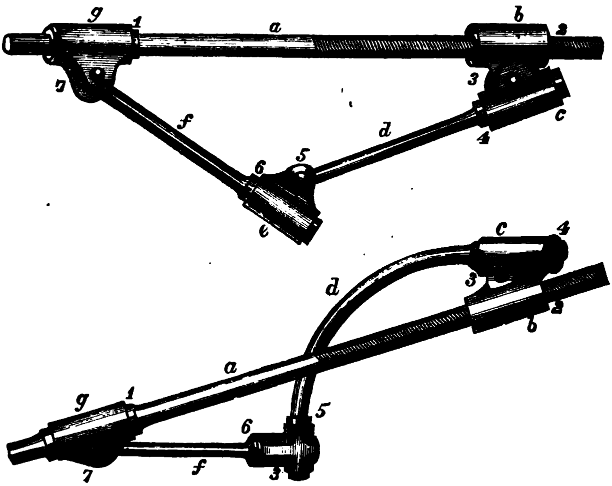


FIG. 325.

We may here summarise briefly the conditions of motion which we have found to exist in the elements of mechanisms and in the mechanisms themselves. The **turning pair**, first of all, gives us a simple rotation about a fixed and permanent (p. 47) axis at a finite distance, and we saw that the motion of the sliding pair was merely the special case in which the axis of rotation (equally fixed and per-

manent) was at infinity. The **screw pair** gives us motion about a fixed and permanent twist-axis, the magnitude of the twist being also constant. It might be considered to include the former pairs as special cases, the one (turning pair) where the pitch of the twist had become zero, and the other (sliding pair) where the rotation of the twist had become zero. Passing from elements to chains or mechanisms, we find that for mechanisms having **plane motion**, the virtual motion of every link is a rotation about a fixed (permanent or instantaneous) axis, at a finite distance or at infinity. In this case, also, all the virtual axes are parallel, so that all planes parallel to the plane of motion cut the axode in similar and equal curves or centrodes, and we can always substitute any one of these curves for the axode (that is, deal with the virtual centre instead of the virtual axis), without impairing the accuracy or completeness of our solutions. In the case of mechanisms having **spheric motion**, the virtual motion is still a rotation about an axis, but the axodes are cones instead of cylinders. Plane sections of these axodes are not, in general, of any value to us. A pair of axodes for the relative motions of any two bodies have a common vertex, and represent the motion of the bodies by rolling on one another,[1] exactly as do the cylindric axodes in the former case. Any sphere which has its centre at the vertex of such a pair of axodes, cuts them both in a pair of spheric sections which roll upon one another as the bodies move, and which touch each other in a point (as S in Fig. 274), which determines, along with the centre of the sphere, the virtual axis. This point of contact is not, however, a virtual centre, as the bodies do not, virtually or otherwise, rotate about it, and these spheric sections of

[1] The proof of the rolling is exactly the same as that given in § 9 for plane centrodes, and does not need to be written out again in full.

the conic axodes are not, therefore, really centrodes. No virtual centres, in the sense in which we have defined these points, exist for spheric motions, and the sets of lines each containing three virtual centres are replaced by sets of planes each containing three virtual axes. With **general screw mechanisms**, lastly, neither virtual axis nor centre exist, the virtual motion is no longer a simple rotation of any kind, but as twist is already reduced to its lowest terms. It may be noticed that in dealing with twist as we did in § 62, looking separately at its two components, rotation and sliding, we virtually resolved it into a pair of rotations, one about the axis of the screw, and the other about an axis at right angles to it and in the same plane, but at an infinite distance. No particular convenience, however, for our purposes, comes from this way of looking at the matter. We have found three different cases of screw motion to occur ino ur work. The regular twist of the screw pair is the first, where the twist-axis is permanent and where the magnitude of the twist is constant. The cases (skew wheels and screw wheels) examined in §§ 68 and 69 come next, and bear the same relation to the screw pair that the motions of a spur-wheel chain (omitting all consideration of the teeth in both cases) bear to those of a turning pair. The twist-axis, namely (for one or more pairs of links), becomes an instantaneous instead of a permanent axis. But the magnitude of the twist is constant, so that the hyperboloids[1] being given, and the value of the twist, the motions are as fully, if not quite as simply, determined as those of rotating bodies whose centrodes are known. This determinateness has nothing to do with the hyperboloidal form of the twist-

[1] It will be remembered that it is these surfaces, the twist-axodes, and not the helical surfaces or their base cylinders, which really determine the motion of screw wheels (p. 555).

axodes, but depends on the constancy of the twist. So long as this condition exists, the relative screw motion of two bodies may be geometrically represented by the loci of their twist-axes as completely as the relative plane motion of two bodies can be by their centrodes. In the third case of general screw mechanisms, however, the case specially dealt with at the commencement of this section, the value of the twist differs with each twist-axis, and varies quite independently of the change of position of the axis. The motions of the mechanism do not seem, in this case, to be determinate by aid of the twist axodes alone, geometrically, but to require also for their determination some expression for the rate of change of the magnitude of the twist itself. It is perhaps fortunate for engineers that problems of this kind have not yet made their appearance in practical work.

CHAPTER XII.

FRICTION IN MECHANISMS AND MACHINES.

§ 71.—FRICTION.

WHEN two surfaces are pressed together it is found that one cannot be moved along and relatively to the other, without the exertion of some definite effort. The resistance, to balance which this effort has to be exerted, is called the **friction** between the surfaces. It can be measured as a force acting from one surface to the other in the direction

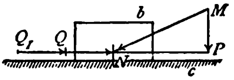


FIG. 326.

of their relative motion, and with a sense such as to offer resistance to that motion. On this account it is sometimes loosely spoken of, without sufficient qualification, as a force which tends always to oppose and never to produce motion.[1] Let *b* and *c* (Fig. 326) be two bodies touching one another, and let *b* slide upon *c* (supposed fixed) under the action of the

[1] A proposition somewhat fiercely attacked by Reuleaux, *Kinematics of Machinery*, p. 594.

force MN. This force has a component MP pressing the surfaces together and causing friction, and a component PN in the direction of motion. Let the external resistance to the motion of b, independently of friction, be QN. Then, disregarding friction, the body b will be receiving an acceleration of $\frac{PN-QN}{m}$ foot-seconds per second, its mass being supposed to be m. If now Q_1Q be the frictional resistance produced by the pressure component MP, the body will be receiving an acceleration only of $\frac{PN-(QN+Q_1Q)}{m}$ foot-seconds per second, and the difference between these two values is the acceleration caused by the frictional resistance. If $Q_1N=PN$ the body b will be moving with a uniform velocity, just as it would do if the friction were absent and QN were equal to PN. We do not say in such a case that the resistance QN does not produce motion; we treat it, on the contrary, as a force in every respect similar to the effort PN, but differing from it in sense. There seems no really sufficient reason for treating frictional resistances in any different manner.

A frictional resistance has always a sense opposite to that of the relative motion of the bodies between which it acts. So far, therefore, as the motion of these bodies relatively to each other is concerned, the acceleration produced by it is always negative. But it is often utilised in order to produce positive acceleration of one of the bodies relatively to a third. Fig. 327 illustrates this, where c is not itself a fixed body, but one capable of sliding upon a fixed body d. Suppose that b were fixed to c by bolts whose united resistance to fracture was RN. Then if the effort could exceed this value the bolts would shear and b would move upon c. But so long as the effort PN is less than RN, the two bodies could not move

relatively to each other, and PN would be balanced by a stress in the bolts (that is, by a *portion* of RN), exactly equal to itself, as QN. The stress in the bolts is what may be called a **derived force**, which has a maximum value RN, but whose actual value is any magnitude less than this which is necessary to balance the external force opposed to it, here PN. This precisely represents the conditions of the case if we substitute frictional resistance between the surfaces for the shearing resistance of the supposed bolts. The friction is a derived force depending here upon the pressure MP, and upon the state of the surfaces. It has some maximum value (which we may suppose to be RN) entirely independent of PN. If PN exceed that value the bodies

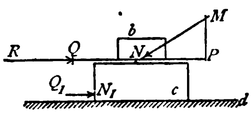


FIG. 327.

will move relatively to each other as in Fig. 326. If PN falls short of that value it will be balanced by a frictional resistance exactly equal to itself, the rest of the possible frictional resistance being non-existent in the same sense as the balance of possible stress in the bolts. Hence if RN be in this case the maximum possible friction, and PN the only driving effort, b will remain stationary relatively to c, under the equal and opposite forces PN and QN. But if c can move relatively to d under some sufficiently small resistance (frictional or other), Q_1N_1, it will be set in motion, receiving the acceleration $\frac{PN - Q_1N_1}{m}$ foot-seconds per second, just as before. In this case it seems legitimate to say that friction is the cause of *positive* acceleration and not negative, for it

is the friction between *b* and *c* which transfers the driving effort *PN* from *b* to *c*, and which in that sense is the cause of the positive acceleration of *c* relatively to *d*.

Both the cases described occur continually in machinery. Wherever two surfaces have to be rubbed together (as in every pin joint or other pair of elements throughout the whole machine), frictional resistances cause negative acceleration, work is expended in overcoming them, they diminish the efficiency of the machine, and it is the object of the engineer to reduce them to the furthest possible extent. In all belt and rope gearing, however (and in a few other cases), the frictional resistance between two bodies (the belt and the pulley) is utilised as the sole means of giving motion to one of them (relatively to a third) and transmitting work to it. It is essential for this purpose that the possible friction between the two bodies (as *b* and *c* in Fig. 327) should be as *great* as possible, and its magnitude only affects the efficiency of the machine if it is too small, so as to permit the relative motion of the bodies which it is intended to prevent.

Experiments made upon the friction of bodies caused to slide upon one another without any, or with little, lubrication, at very moderate velocities, and with small intensities of pressure,[1] have established the facts that under these conditions the friction is independent of the area of contact and intensity of pressure, and is practically independent of the velocity of rubbing, being for any given pair of surfaces proportional simply to the total normal pressure. Under such conditions, therefore, the frictional resistance can be found at once for any known value of the pressure *P* by multiplying it by some co-efficient μ dependent essentially on the

[1] By *intensity* of pressure is meant, as formerly, pressure per unit of area.

nature of the surfaces, so that the value of the friction is written

$$F = \mu P.$$

The multiplier μ is called the **co-efficient of friction**, and is assumed to be fairly constant for given materials with such surfaces as are commonly used.

Engineers, however, have seldom to do with unlubricated rubbing surfaces, and they have to deal with surfaces moving often with very high velocities, and under very great and frequently varying pressures. Under these conditions the "laws" of friction, as they have just been stated, not only do not hold exactly true, but fail even to represent approximately the more complex phenomena with which they have to deal. At many speeds and loads which are of daily occurrence in machinery, velocity and intensity of pressure have an enormous effect on the friction, and not only these, but the temperature of the surfaces and the nature of the lubricant. The nature of the rubbing contact also, whether continuously in one sense or continually reversed, whether the surfaces be flat as in a guide, or cylindrical as in a bearing, whether contact exist throughout a surface or only along a line, very greatly affects the friction. The actual material of which the surfaces consist forms only one out of an immense number of conditions which determine friction under a given load. In fact, although all the rubbing surfaces in a machine were made of the same material, and had as nearly as possible the same smoothness, the co-efficient of friction, that is the quantity by which the total pressure on each surface would have to be multiplied to find the friction, instead of being practically constant, might be ten times [1] as great for some pairs of surfaces as for others. In

[1] Often enormously more than ten times. The particular number ten is not intended to have any special significance.

each particular pair of surfaces, with its own special area, velocity, form, amount of lubrication, and so on, the frictional resistance bears a different proportion to the load, and can be estimated from it only by the use of a different co-efficient. Under these circumstances it is perhaps misleading to retain the much-used phrase, "co-efficient of friction," for this inconstant multiplier. The co-efficient of friction has the certain definite meaning which has already been explained, and which limits its use to solid friction under certain simple conditions. It is so thoroughly associated with the idea that friction is proportional to load, that it seems unadvisable to call by its name a mere multiplier which may even itself vary inversely as the load. We shall, therefore, speak rather, in the following sections, of the **friction-factor** for a given pair of surfaces, meaning by this expression simply the ratio, dependent on all the varying conditions already mentioned, of the frictional resistance of those surfaces to the pressure causing it. We may, therefore, write $\frac{F}{P}=f$, the friction-factor, so that we still have

$$F=fP$$

but with the condition that f is a quantity whose value has to be separately considered for each set of conditions.

In every mechanical combination, from a pair of elements to a machine, some effort is at each instant expended in balancing friction, some work therefore is done, as the machine moves, merely in overcoming frictional resistance. If we call the remaining effort or work, as the case may be, the *nett* or *useful* effort or work of the combination, the ratio $\frac{\text{useful effort}}{\text{total effort}}$ or $\frac{\text{useful work}}{\text{total work}}$ is called the **efficiency** of the apparatus. Where the ratio is between amounts

of *effort* only, we have the efficiency simply at the instant and for the position at which the effort has been measured. Where the ratio is between quantities of *work*, it gives us the *average* efficiency during the period in which that work has been done.[1] The reciprocal of the efficiency was called by Rankine the **counter-efficiency**, a name of great value. The counter-efficiency expresses, of course, the ratio in which it is necessary that the whole effort exerted or work done should exceed the nett value of the effort or work required.

The older results as to friction rest mainly on the experiments of Morin (dating as far back as 1831), which were most carefully conducted, and the results of which, within the limits and under the conditions to which they are fairly applicable, there is no reason whatever to doubt.[2] But they were made under conditions which, however well they may represent those of ideal solid friction (as they were intended to do), do not at all represent those of ordinary machinery. In spite of the numerous experiments of Thurston and others, of the brake trials of Westinghouse and Galton, and of the recent experiments of a Research Committee of the Institution of Mechanical Engineers made by Mr. Tower, we have still not

[1] The idea of the efficiency of a machine, now so familiar, we appear to owe to Moseley (*Phil. Trans.* 1841, and *Mech. Principles of Engineering*, 1843). He called it the *modulus* of the machine, and worked out its value in an immense number of different cases, including that of toothed gearing, spur and bevel.

[2] For Morin's experiments on sliding friction, see the *Mémoires de l'Institute* for 1833, which contain two series. The friction was measured between blocks of various materials and flat rails of the same or of different materials. The blocks were heavily loaded, and the motion (having been first started by special apparatus) was kept up by the pull of a descending weight. In some cases the velocity was uniform, in most accelerated. In no case could the experiment last more than a few seconds. The velocity and the amount of the pull were registered automatically. The distance through which sliding occurred was from ten to twelve feet. Morin's experiments on *Frottement des axes de rotation* were made in 1834.

nearly sufficient information to enable us to give probable values for the friction-factor under many of the most important cases occurring in practice.

Let us take first friction in journals or pin joints generally, assuming that the one surface moves continuously over the other, and does not reciprocate. What we call the "pressure on the bearing" does not here represent the actual pressures between the surfaces, but rather the total value of the components of those pressures in a certain direction. As it is only this nominal total pressure that forms part of our data in practice, it is sufficient to accept it as a starting point, without troubling ourselves here as to the real distribution of pressure. Engineers often speak of the pressure *per square inch* upon a bearing, by which they invariably mean the total pressure divided by the area of the bearing upon a plane normal to it, that is, by the product of the length of bearing and diameter of shaft. This nominal "pressure per square inch" is, therefore, an entirely conventional unit. Tower's experiments, which were made upon a steel journal four inches diameter and six inches long, give the remarkable result that for a given speed the total friction remains nearly constant for all ordinary loads[1] not too great nor too small for the particular lubricant used, so long as the lubrication was kept "perfect." The friction-factor, therefore, varied *inversely as the load.* It varied at the same time directly (very nearly) as the square root of the velocity. The formula

$$f = 20\,c\frac{\sqrt{v}}{P}$$

expresses very closely the results of these experiments (so long as the lubrication was kept perfect) for a temperature

[1] These experiments did not go below 100 pounds per square inch nominal. *Proc. Inst. M. Engs.* 1883 and 1884.

of 90° Fahr. v is the peripheral velocity of the bearing in feet per minute, and P the nominal pressure per square inch upon it. c is a co-efficient depending on the lubricant, and has a value of ·0014 for sperm oil (up to 300 pounds per square inch pressure), ·0015 for rape oil, and ·0018 for mineral oil (up to about 450 pounds per square inch), ·0019 for olive oil (up to 520 pounds per square inch), and about ·003 for mineral grease (between 150 and 625 pounds per square inch). The value of $20\frac{\sqrt{v}}{P}$ is unity (nearly) at a speed of 250 feet per minute and a pressure of about 310 pounds per square inch, so that c is itself the friction-factor, or co-efficient of friction, for these conditions. It will be noticed how much smaller it is than the value usually taken.

When the lubrication was not made "perfect" by the use of an oil bath, but the oil supplied, as regularly as was possible, by a syphon lubricator, the friction-factor was about four times as great as that given. It followed the same law as to variation inversely as the pressure, but its variation with velocity was much less than before, and was irregular. When the lubrication was reduced to a minimum ("so that the oiliness was only just perceptible to the touch"), it was increasingly difficult to get uniform results, but those that were obtained approximated distinctly, as was to be expected, to the usually assumed conditions of solid friction. Between loads of 100 and 200 pounds per square inch the friction-factor diminished as the load increased, but much less rapidly, and from 200 to 300 pounds per square inch (at which pressure "seizure" occurred), the factor remained nearly constant, forming a real co-efficient of friction varying only from ·008 to ·010. The variation with the velocity was larger at the lowest pressures, but smaller and irregular afterwards.

Temperature was found to affect the friction very greatly. With a load of 100 pounds per square inch, for instance, the friction with lard oil was about double as much at 75° as at 120°, and about three times as much at 60° as at 120°. No doubt there is a best possible temperature for each lubricant at each load, namely, that temperature which keeps it as thin as possible without making it so liquid as to be squeezed out.

Mr. Tower seems to have shown beyond doubt that with perfectly lubricated journals the metal surfaces should be, and are, separated by a film of lubricant,[1] and this fact at once explains the immense discrepancies between the results just stated and those obtained in such experiments as Morin's.

Nominal pressures of 200 to 500 pounds per square inch are common in the bearings of machinery, but in certain cases, such for instance as the pin in a piston rod, head pressures of 800 to 1,200 pounds per square inch are constantly used without any ill effects. In these cases, however, the speed of rubbing is slow[2] (quite possibly slower than that to which the friction diminishes with the velocity), the motion is reciprocating, and above all the pressure is alternating in direction, first on one side of the pin, then on the other. We have no experiments on friction under these conditions, but know by experience that bearings working in this fashion can carry a very much greater load than those loaded in one direction and revolving continuously under the load. The crank pin of an engine

[1] *Proc. Inst. M. Engs.*, 1883, etc.

[2] At *excessively* slow speeds the experiments of Jenkin and Ewing have shown that friction increases as the velocity diminishes, until (probably) the friction of motion (with which only we concern ourselves here) merges continuously into the friction of rest. But these speeds are much slower than any we have to deal with.

forms an intermediate case, the velocity of rubbing may be about the same as for the shaft, but the pressure is alternately in opposite directions. In the crank shaft of an engine, and still more in ordinary shafting, the weight of fly wheel, pull of belts, etc., cause the general direction of pressure to remain comparatively unaltered. Correspondingly the pressure in such cases is made considerably less than in a crank pin, although the velocity of rubbing is about the same.

The friction in the ordinary pin joints of linkwork, where the lubrication is not so well attended to as in shaft bearings, must vary enormously. As long as the lubrication is uniform, even if it is very small, it ought to be possible to work them with a friction-factor of ·010 to ·015, remaining approximately constant at such loads as they can carry. With freer lubrication a value for the friction-factor of

$$f = 0{\cdot}015\,\frac{100}{P} = \frac{1{\cdot}5}{P}$$

may approximately represent what can be obtained.

As to the friction of such lubricated flat surfaces as guide blocks, there appear to be no modern experiments. Those of Morin, already mentioned, give for sliding metal blocks, with "lubricant constantly renewed," a true co-efficient of friction of 0·05,[1] varying neither with the velocity nor with the pressure. This was at various velocities and at pressures averaging 28 pounds per square inch, and sometimes as much as 110 pounds. Ordinary steam-engine guide-blocks have a velocity (in alternate directions) varying in each stroke from 0 to 500 or more feet per minute, and under these circumstances it is found that they work best when the maximum pressure upon them is kept under 40 or

[1] This figure is given by Morin himself in connection with his 1834 experiments, as representing his very best possible results, but it is considerably lower than those results themselves, in which, no doubt, the lubrication was very imperfect.

50 pounds per square inch, which points to a friction-factor very much higher than for bearings.

On the friction of pivots there is very little experimental evidence. The case is complicated by the fact that the velocity of rubbing varies from zero to a maximum over the surface, from the centre outwards, and that the distribution of pressure varies also (see § 75) in a way which we do not know. Here, as in cross-head pins, it is found practically possible to allow often a larger pressure than in ordinary bearings, and an average of 700 and 800 pounds per square inch, and in some cases double as much, can be carried without injury.

As to the friction-factor for the rubbing in higher pairs, such as wheel teeth, there is also exceedingly little experimental evidence. The brake experiments of Westinghouse and Galton [1] show that at equal velocities and pressures the friction-factor for the wheels skidding on the rails was only about one-third as great as for the wheels rubbing on the brakes. The comparison is between rubbing with *line* contact (as with higher pairs) and rubbing with *surface* contact. If these results are applicable to the lubricated, or semi-lubricated, higher pairings which occur in machinery, their friction-factor must be much lower than it would otherwise be assumed to be, and this is probably the case.[2] If the smallness of the surface causes a reduction of the friction where there is no lubrication, it is probable that it may cause still more where even imperfect lubrication exists.

In the case of higher pairing, as with toothed wheels, cams, &c., we have seen (§ 18) that the relative motion of the surfaces is not pure sliding, but is equivalent to a combination of rolling and sliding, the particular lines which are in

[1] *Proc. Inst. M. Engs.* 1878 and 1879.

[2] This conclusion is strongly corroborated by recent experiments by Mr. John Goodman in the author's laboratory at University College.

contact changing continuously, not only on one but on both surfaces. There is no evidence to show whether this affects the frictional resistance. The "rolling friction" which may occur with it is entirely negligible in comparison to the other quantities.

The frictional resistance of belts or straps upon pulleys is further mentioned in § 78, and the resistance of ropes and straps to bending, which plays an important part in determining the counter-efficiency of pulley tackle, is looked at in § 80. Of the friction in metallic-packed pistons we know very little. In the case of the steam engine much of the work converted into heat here may possibly be recovered through the steam. Some experiments of the author's give 0·80 to 0·85 as the efficiency of transmission through a hemp-packed hydraulic stuffing-box, in ordinary working conditions, at pressures from 250 to 700 lbs. per square inch. The resistance experienced when one surface is made to *roll* upon another, which is often called "rolling friction," and which was first fully examined, probably, by Osborne Reynolds,[1] is so small, and so seldom requires consideration in machinery, that we do not examine it here.

Before going on to examine quantitatively the frictional efficiency of machines, it is well to point out that there are very few cases in which the forces causing friction are at all completely known, and very few cases, therefore, in which we are able to find (apart altogether from our imperfect knowledge of the friction-factors) the real total efficiency of a machine. In a steam engine, for instance, we can find without difficulty the efficiency of the machine (within such limits of accuracy as we know the values of f) so far as it

[1] *Phil. Trans.* vol. clxvi. See also Cotterill's *Applied Mechanics*, Art. cxxii.

depends on the steam pressure, or on the velocities and the weights of the various bodies. But apart altogether from the pressures produced by, or dependent on, these things, there is friction in the machine. It cannot be turned round, empty, without the exertion of effort. This additional friction arises from tightness of "fit" of the various parts, and from resistances due to tightened-up brasses, &c. The magnitude of the pressures caused by fit and by screwing up nuts is practically altogether unknown in any particular case, and the loss of efficiency from these causes remains determinable only by experiment, and may often be very large indeed.[1]

§ 72.—FRICTION IN SLIDING PAIRS.

LET a and b be portions of the two elements of a sliding pair, and let a be acted upon by any force $p = MN$ in its plane of motion. To find the force in the direction of motion which p can balance, we formerly (p. 276) simply resolved p in that direction and normal to it, as PN and MP, the latter component passing through the virtual centre. The normal component we formerly neglected; it was sufficient for us to know that it was internally balanced by

[1] Since this chapter was in type a very important experimental paper on the subject of the efficiency of spur, skew, and worm gearing has been published. The paper is by Mr. Wilfrid Lewis, and was written for the American Society of Mechanical Engineers (Boston meeting, 1885); it will be found *in extenso* in *Engineering*, vol. xli., pp. 285, 363, and 581. The results given in it are by no means as concordant as could be wished, and their serious defects are freely admitted by the author of the paper, but as the experiments are practically the only ones of the kind available, they are, in spite of all drawbacks, most valuable. They show that the efficiency is not affected much by amount of pressure (contact was always line or point contact, it will be remembered), that it increases rapidly with the velocity, that it increases regularly as the skew of the teeth diminishes, and that within the limits of experiment and of experimental error it was the same for skew as for

stresses within the material (p. 261) and could not cause any alteration in the direction of motion. It is this component which we have now to attend to. Being at right angles to the direction of motion it must be at right angles also to the surfaces of contact, for by hypothesis these must be parallel to the direction of motion. The hitherto neglected

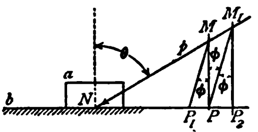


FIG. 328.

component through the virtual centre, then, is a normal pressure between the surfaces of the elements, and as such produces friction between them. If f be the friction-factor for the given conditions, then the frictional resistance due to the pressure MP is $f.MP$. This resistance lessens, by

screw gearing. The following table, compiled from a diagram, gives some of the most important of Mr. Lewis's results. These efficiencies appear to include the loss by friction in the pivot or thrust bearings of the worm and screw gearing. Mr. Lewis gives no particulars as to the system of lubrication used ; apparently it was somewhat imperfect.

DESCRIPTION OF GEARING.	VELOCITY AT PITCH LINE IN FEET PER MINUTE.				
	10	50	100	150	200
	EFFICIENCY.				
Spur wheel and pinion	0·940	0·972	0·980	0·984	0·986
Screw do. (45°)	0·870	0·935	0·955	0·963	0·966
Do. do. (30°)	0·815	0·900	0·930	0·941	0·947
Do. do. (20°)	0·748	0·855	0·900	0·916	0·924
Do. do. (15°)	0·700	0·820	0·872	0·893	0·902
Do. do. or worm (10°)	0·615	0·760	0·820	0·848	0·862
Do. do. } do. (7°)	0·534	0·695	0·765	0·799	0·815
Do. do. } do. (5°)	0·445	0·620	0·700	0·736	0·761

an amount precisely equal to its own magnitude, the external resistance which can be balanced by MN. Thus if we set off $P_1P = f.MP$, or what is often more convenient make the angle P_1MP such that its tangent is equal to the friction-factor, we have at once NP_1 as the nett external resistance required to balance MN, or against which MN can cause motion. An angle of such magnitude that its tangent is equal to the friction-factor we call the **angle of friction**, and denote by the letter ϕ.

It is sometimes said, loosely, that whereas when there is no friction the pressure between the surfaces is normal to them, when there *is* friction the pressure ceases to be normal, but is inclined to the normal at the angle of friction.[1] This is not strictly true. The pressure from surface to surface is always normal to the surfaces—it cannot be otherwise. The *sum of the pressure and of the frictional resistance due to it* may be rightly described as making the angle ϕ with the normal to the surfaces, and this is no doubt what is meant, although imperfectly expressed, by the statement just quoted. In the figure MP remains in magnitude and direction the pressure between a and b whether or not there be any friction. $(MP + PP_1) = MP_1$ is the sum of the pressure and the friction, and this line is inclined at the angle ϕ to MP, *i.e.* to the normal to the surfaces. The triangle of forces MNP, which formerly represented *three* forces only,—the effort MN, the external resistance NP, and the pressure PM,—now represents *four* forces, the effort MN, the external resistance NP_1, the frictional resistance P_1P, and the pressure PM. In taking the triangle MNP_1 to represent the forces in action, that is, in resolving MN in the directions NP and P_1M instead of NP and PM, it must be

[1] The general theorem was given first by Moseley, *Camb. Phil. Trans.* 1834.

always borne in mind that one of the sides of this triangle (here P_1M) represents the sum of two forces, and not a single force merely. Following Dr. Lodge it may be convenient to call this sum the **total reaction** between the surfaces.

The *efficiency* of the combination is represented by the ratio $\frac{NP_1}{NP}$, the *counter-efficiency* by $\frac{NP}{NP_1}$.* If the problem had been given in the reverse direction, and we had had to find the force necessary to move the block against a given nett resistance NP, we should simply have had to draw

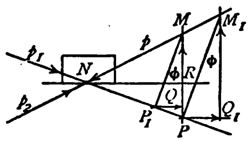


FIG. 329.

PM_1 making the angle ϕ with the normal PM, and so have obtained M_1N as the required effort instead of MN. It is hardly necessary to point out that $M_1N = MN \times$ counter-efficiency, *i.e.* $\frac{M_1N}{MN} = \frac{NP}{NP_1}$, nor that in this last case M_1P_2 is the normal pressure between the surfaces, and PP_2 the frictional resistance produced by it. The point of *a* and *b* which is supposed to be the centre of pressure throughout is the point N.

If the resistance were not in the direction of motion, but in any such direction as p_1 (Fig. 329), we can still apply exactly the same constructions. NP represents the external

* Professor R. H. Smith calls the fraction $\left(1 - \frac{NP_1}{NP}\right)$ or $\frac{P_1P}{NP}$, the *inefficiency* (see p. 591).

resistance balanced by p without friction, NP_1 with friction. In the latter case MQ is the normal pressure between the surfaces, in the former MP. The frictional resistance is P_1Q (*not* P_1P) and the reaction, or sum of P_1Q and QM, makes, as before, the angle ϕ with the normal to the surfaces. The efficiency is $\frac{NP_1}{NP}$. The length M_1N represents, as in the last case, the total effort required to balance a nett external resistance NP. In this case M_1Q_1 would be the pressure, and PQ_1 the friction, the efficiency being $\frac{MN}{M_1N} = \frac{NP_1}{NP}$.

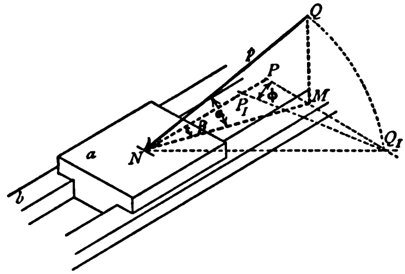


FIG. 330.

Should the resistance be exactly opposite to p (as at p_2), there would, of course, be no friction, and the efficiency would be unity, the points P_1, P, and Q all coming together in M.

Of course it may quite well happen that the driving force does *not* act in the plane of motion (see § 64, p. 512), but obliquely to it as in Fig. 330. Here p is first resolved into MN in the plane of motion, and QM normal to it. This latter component causes friction in the flanges of the block, at right angles to its plane of motion, quite independently

of the friction caused by MP, the normal component of MN (the letters correspond to those of Fig. 328). In estimating the efficiency in such a case as this, assuming the same friction-factor for the block itself and its flanges, it is most convenient to add the two friction-producing components QM and MP together, and then work out in the plane as if Q_1N were the total force instead of MN. The working will give the nett external resistance NP_1 which can be balanced by the force p in addition to the friction caused by it both at the surface of the block and of its flanges.

It will be sufficient for us, in what follows, to take into account only the component of the total force which acts in the plane of motion. If the force itself is oblique to the plane, its friction-causing equivalent in the plane must first be determined in the fashion just given, and this equivalent used instead of the force itself. Practically it will be found that the construction of Fig. 330 gives both more quickly and (unless a specially good protractor be used) more accurately than a calculation, for in almost every case the angles have to be set off first from other data, and their magnitude then measured. But in case the angles are given directly, the value of the efficiency can be found by calculation from the formula

$$\frac{NP_1}{NP} = 1 - \left(\frac{\tan \alpha}{\cos \beta} + \tan \beta\right) \tan \phi,$$

the value of the angles being as marked in the figure. Taken separately the friction in the flanges is

$$(QN \sin \alpha) \tan \phi,$$

and the friction on the surface of the block is

$$(QN \,.\, \cos \alpha \,.\, \sin \beta) \tan \phi.$$

Cases of perhaps greater practical importance arise when the surfaces of contact are not parallel to the plane of motion, although the force may be so. Such cases are

illustrated in Figs. 331 and 332. In each case the direction of motion is supposed to be at right angles to the plane of the paper, and the same force p, $= MP$, acts in that plane. This force may either be by itself, or may be the normal component (as MP in Figs. 329 and 330) of other forces acting on a. In Fig. 331 the surface of the block is at right angles to the pressure, there is no friction in the flanges, and the total frictional resistance (its direction of course being normal to the paper) is represented in magnitude by $PP_1 = p \tan \phi$. In Fig. 332 the surface of the

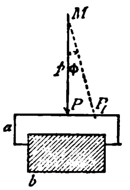


FIG. 331.

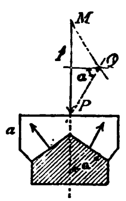


FIG. 332.

block is bevelled or V-shaped. The pressure MP is therefore not balanced by an exactly equal and opposite force, but by forces in the direction shown by the small arrows, normal to the block surfaces. The amount of friction depends on the magnitude of the normal pressures, and is therefore here not proportional to MP, but to the sum of its components MQ and QP in the given directions. Each of these components is equal to $\frac{p}{2} \times \frac{1}{\sin \alpha}$, so that together they are equal to $\frac{p}{\sin \alpha}$, and the whole frictional resistance is $\left(\frac{p}{\sin \alpha}\right) \tan \phi$. The friction in a bevelled block, therefore, whose vertex angle is 2α, is greater than that in a flat block,

other things being equal, in the ratio $\frac{1}{\sin \alpha} : 1$. It is therefore *unadvisable* to use bevelled surfaces for sliding pairs in any cases (as piston and slide-rod guides, etc.) where friction is disadvantageous, and *advisable* to use them in any cases (as brake blocks, rope-pulleys, etc.) where it is desired to make the friction as great as possible.

The friction in a wedge is covered by the case of Fig. 332. The wedge in itself represents only a couple of bevelled surfaces like *b* above, but with the angle α very small, so that the frictional resistance is very large.

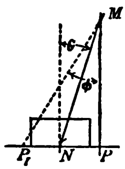


FIG. 333.

The condition mentioned in § 71, that frictional resistance should render motion impossible, is reached when the effort acts so as to make a smaller angle than ϕ with the normal to the surfaces. Thus in Fig. 333 the angle θ is less than ϕ. The pressure MN is capable of causing friction as great as P_1P, but cannot supply more than the effort PN to cause motion.[1] The ratio $\frac{P_1P}{PN}$ is constant, and independent of the magnitude of MN. In such a case, therefore, no force, however great, in the direction of MN, can cause the one surface to slide on the other.

[1] See p. 567, § 71.

§ 73.—FRICTION IN TURNING PAIRS.

THE efficiency of a pin joint, or turning pair, is generally very much greater than that of a sliding pair; the way in which the equilibrium of forces is affected by friction is, however, a little more troublesome to understand. The construction which we shall use is that first given by Rankine, and afterwards much developed in its uses by Jenkin, Hermann, and others. Let the circle in Fig. 334 show one element a of a turning pair, its radius being OR. Let f and f_1 be the directions respectively of the effort and the resistance to the motion of a, its sense of rotation being indicated by the arrow, and let f be represented in magnitude by MN. Then apart from friction we know that NP will be the resistance balanced by f, and PM the radial pressure,[1] which will act between the pin a and the eye at the point R. The friction at R must, as a force external to a, act in the opposite sense to that in which R is moving, as shown by the arrow f_2. The direction of the reaction must make the friction angle ϕ with the normal to the surfaces at the centre of pressure, or point at which we may assume the pressure to be concentrated. But at the same time the reaction, or sum of the friction and the normal pressure, must have such a direction that it will pass through N, or the system of forces will not be in equilibrium. The centre of pressure cannot, therefore, be any longer at R, but must be at R_1, a point such that its radius R_1O makes the angle ϕ with the line R_1N joining it to N. The resultant pressure between the

[1] This may be more fully described as the sum of the pressures between the surfaces, in the same sense as MP in Fig. 332 is the sum of the pressures on the V block. The actual distribution of true radial pressure we do not know, but it must bear the same relation to PM in all bearings in which there is the same arc of contact. See p. 572, § 71.

surfaces is still radial (*i.e.* normal to the surfaces), but its position is R_1O instead of RO. The sum of friction and pressure has the direction R_1N. If we now draw MP_1 parallel to R_1N, we find at once, exactly as in the last section, the nett resistance NP_1, which the given effort can balance in addition to the friction in the bearing. The efficiency is, as before, equal to $\frac{NP_1}{NP}$.

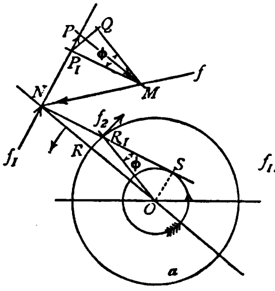


FIG. 334.

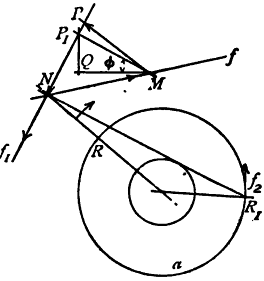


FIG. 335.

By resolving MP_1 into the components P_1Q parallel to f_2, and QM parallel to R_1O, we have in them the magnitudes of the friction and the total normal pressure respectively. (The angle QMP_1 is equal to ϕ by construction.) The forces under which a is in equilibrium are represented by the four sides of the closed polygon MNP_1QM. If our only object in the construction is to find NP_1, or the efficiency of the combination, we draw only the triangle MNP_1, but it must not be forgotten that (just as in the last section) P_1M is the sum of two distinct forces, and in particular that it does not represent, either in magnitude or in direction, the pressure between the surfaces of the pin and eye, or shaft and bearing.

We have now to see how the point R_1 can be practically found. Let OR, the radius of the shaft, $=r$, and let there be a small circle drawn about O with radius $OS = r \sin \phi$.[1] Then R_1 can be found at once as the point in which a line drawn through N touching this small circle, cuts the pin circle. Obviously the angle OR_1S is equal to ϕ, for its sine is $\frac{OS}{r}$, and this is already made equal to $\sin \phi$ by construction. The small circle used in this construction has received the name of **friction circle**, and as such we shall refer to it.

If the sense of the effort were reversed, so as to be NM instead of MN, the construction would have been that of Fig. 335, in which the same lettering is used as in the last figure. Here the sense of rotation is reversed, but the sense of the normal pressure is reversed also (PM changed to MP), so that the pressure comes on the lower side of the bearing instead of the upper, and the friction, still opposing the motion of a, again acts from left to right.

If, however, instead of changing the sense of the effort in Fig. 334, we had reversed the sense of rotation by making NP the effort and MN the direction of the resistance, the line NR_1S would have changed its position to the left instead of the right of NR, the sense of the friction being reversed, while it acts still on the same side of the shaft. There need never be any hesitation as to whether NS lies to right or left of NO. The points N, M and P are always given, and the point P_1 must always lie between N and P, for $\frac{NP_1}{NP}$, which is equal to the efficiency, must always be less than

[1] Under ordinary circumstances ϕ itself is not given—only $\tan \phi$, the friction-factor. For any such small angle $\tan \phi$ may be taken as equal to $\sin \phi$, and the radius OS made $= r \tan \phi$, which will save some trouble.

unity. It being thus known how MP_1 lies with reference to MP, the relative slopes of their parallels NS and NO are also known. In more complex cases, where this cannot so readily be done, find the sense of the pressure, as a force external to the body or element which we treat as the moving one, and draw it so as to oppose, with this sense, the rotation of the friction circle considered as a part of the moving body. This rule, which must be thoroughly mastered, finds many applications in § 77.

It will be noticed at once that in the figure, in order to make the construction as clear as possible, the angle ϕ has been taken absurdly large. In ordinary circumstances, as we have seen in § 71, it is exceedingly small, and the efficiency is, therefore, exceedingly near to unity. If the point N were upon the periphery of the shaft or at an equal radius, that is, if it coincided with R, the construction would become identical with that given for a sliding pair. R_1 would coincide with R, the angle ONS would be equal to ϕ, and the friction circle would be superfluous. In that case the efficiency of the turning pair would be exactly equal to that of a sliding pair working with the same friction-factor. As the point N, however, is moved further away, the angle at N becomes smaller than ϕ and continually diminishes, although the angle OR_1S remains constant. Thus as the radius of N increases the angle PMP_1 diminishes, and the efficiency becomes greater than that of a sliding pair, becoming more and more nearly unity as N goes further and further off. This corresponds, of course, to the fact that the radius of the friction is constant (and equal to the peripheral radius of the shaft), while the radius of the effort is constantly enlarged, so that a diminishing fraction of it is required at the constant radius of the friction. If in any combination, and the case is quite a possible one, the

driving effort should act at a *smaller* radius than that of the surfaces in contact, the efficiency of the combination would be *less* than that of a sliding pair under the same forces, and the efficiency would diminish, as NO was diminished, until it became zero, the counter efficiency increasing at the same time to infinity, so that the pin would be immovable by any force whatever in the given direction with the given value of ϕ.

There is often *axial* friction in a turning pair as well as the pin friction just considered, due to some component of effort or resistance normal to the direction of motion and therefore parallel to the axis of the shaft. Where this occurs it has to be considered separately: it will be found discussed under the head of *friction in pivots* in § 75.

The *moment* of the frictional resistance in a bearing often requires notice, and also the *work* done in overcoming the friction. The former is simply $P_1Q \times RO$, (or $QP_1 \times RO$ as the case may be), and the work done per minute is

$$2\,\pi\,n\,P_1Q.\,RO.^* \text{ foot-pounds,}$$

where n is the number of revolutions per minute.

One very important point still remains in regard to the construction of Figs. 334 and 335. Under ordinary circumstances the value of the friction factor is so small that the friction circle cannot conveniently be drawn with reasonable accuracy. If this is the case, and we have to deal with a single turning pair only, it is of course probable that the loss of efficiency due to friction is so small that it may be neglected. But even where the coefficient of friction is too small to be conveniently handled in the way described in this section, it may still happen that in consequence of the number of pin joints in a mechanism, and of the special

* To obtain work in foot-pounds RO must of course be measured in feet.

directions of the forces acting upon the links connecting them, the total loss of efficiency is too great to be neglected. Professor R. H. Smith has for several years past used a method, in these cases, which seems to meet the difficulty satisfactorily. He points out that if the friction circle, or any number of friction circles corresponding to a mechanism (as in § 77), be increased in diameter in a ratio n, although the lines of the forces and pressures will be thereby changed variously in direction, the new value of what he calls the *inefficiency* of the combination (or value of $1-E$) will be changed approximately in the same ratio. Thus if E be the real efficiency of any combination, and $I=(1-E)$ its real inefficiency, then if these quantities be calculated by the use of friction circles n times too large, and the quantities E' and I' obtained by this calculation, the value of the new inefficiency I' or $(1-E')$ will be approximately equal to $n(1-E)$, and the real efficiency can be found approximately by the expression

$$E=1-\frac{(1-E')}{n}=1-\frac{I'}{n}$$

In using this method it must be particularly noticed that it becomes inapplicable altogether (*i.e.* not reasonably approximate) if the friction circles as actually used become at all large, *i.e.* more than $\frac{1}{6}$ or $\frac{1}{8}$ of the diameter of the journal or pin. The degree of approximation varies much with the complexity of the mechanism, the proportions of the links and the directions of the forces, but for practical purposes Professor Smith's method will be found quite sufficiently accurate, and very convenient and useful, so long as the magnified value of the friction-factor is kept (say) not greater than $\frac{1}{8}$. In applying the method to combinations containing both turning and sliding pairs, the value of tan ϕ must be increased for the latter in the same ratio as for the former.

§ 74.—FRICTION IN SCREWS.

THE necessary treatment for a screw pair as regards friction is in the first instance precisely that which we have used for a sliding pair. The surface of the screw thread of *a* slides upon that of the thread in the nut *b* (Figs. 336 and 337). The direction of sliding is taken to be the direction of the tangent to the thread at its mean radius. The resistance PN is taken as axial, the effort NM as at right angles to the axis, and the latter is assumed to be applied

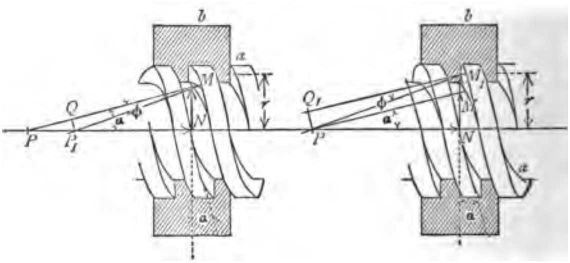


FIG. 336. FIG. 337.

at a radius equal to the mean radius of the screw thread. MP is the direction of pressure between the surfaces, normal to the assumed tangent. *Without* friction NM balances PN, the ratio between them being (as on p. 479), $\frac{NM}{PN} = \tan \alpha = \frac{\text{pitch}}{\text{circumference}}$ of screw thread. *With* friction there is precisely the same change as in the sliding pair. In Fig. 336 the given effort NM will balance only the axial resistance P_1N, the efficiency being $\frac{P_1N}{PN}$ as before. MP_1 is the sum of the friction QP_1 and the normal pressure

MQ, and is inclined at the angle ϕ to the direction of the latter. If the resistance PN is fixed and we have to find the increased effort to balance it, we have it in NM_1, Fig. 337, the ratio $\frac{NM}{NM_1}$ being the efficiency, and equal to the ratio $\frac{P_1N}{PN}$ of the last figure. Q_1P is now the frictional resistance, M_1P the total reaction, and M_1Q_1 the normal pressure between the surfaces.

The great loss by friction in screws, and the small efficiency of a screw pair, is well known and often remarked on. It must not be supposed, however, that this is specifically due to the screw surface or anything connected with it. It is due solely to the particular values of the angles between effort and resistance and direction of motion which happen to be common in screws. A sliding pair working with the same angles would have precisely the same efficiency. It may save misapprehension if this is always clearly borne in mind.

The ratio $\frac{NM}{PN}$ of effort to resistance without friction is equal to tan α. The ratio $\frac{NM_1}{PN}$, with friction, (or the equal ratio $\frac{NM}{P_1N}$), is equal to tan $(\alpha+\phi)$. The value of the efficiency may therefore be stated algebraically as

$$\frac{NM}{NM_1}=\frac{P_1N}{PN}=\frac{\tan\alpha}{\tan(\alpha+\phi)},$$

the counter-efficiency being, of course, the reciprocal of this. The small efficiency of screws arises from the fact that in them the angle α is always a small angle, so that although ϕ is no larger than in a sliding pair, $(\alpha+\phi)$ may be proportionately very much larger than α. Enlargement of α would tend to increase the efficiency, but at the same time might

still faster diminish the "mechanical advantage" of the screw, which is simply the ratio $\frac{NM}{PN}$, numerically equal to tan α.

Two important limiting cases occur. If $\alpha = 0$, tan $\alpha = 0$ and the efficiency $= 0$. The screw becomes simply a series of parallel rings, which we know to represent a screw of zero pitch. If $\alpha = (90° - \phi)$, tan $(\alpha + \phi) = \tan 90° = \infty$, and again the efficiency is zero, although for a different reason. In practice tan α varies commonly from 0·05 to 0·15, going occasionally as high as 0·2. Tan ϕ, the friction-factor, varies more widely. With ordinary screw threads of square section, with surface contact and with no special lubrication it may be 0·1 and even greater.[1] By proper lubrication this will be greatly reduced, and where along with complete lubrication there is point contact (as with well-made worm gearing), the value of tan ϕ may fall as low as 0·01 and possibly less. The following table

TAN α	EFFICIENCY $= \frac{\tan \alpha}{\tan(\alpha + \phi)}$				
	Tan ϕ = 0·01	Tan ϕ = 0·02	Tan ϕ = 0·03	Tan ϕ = 0·10	Tan ϕ = 0·20
0·000	0	0	0	0	0
0·025	0·713	0·555	0·458	0·203	0·112
0·050	0·829	0·706	0·622	0·331	0·196
0·075	0·883	0·741	0·699	0·429	0·270
0·100	0·906	0·828	0·766	0·495	0·325
0·125	0·924	0·858	0·805	0·552	0·376
0·150	0·935	0·877	0·829	0·592	0·414
0·175	0·943	0·893	0·849	0·627	0·450
0·200	0·950	0·904	0·865	0·656	0·480
0·225	0·954	0·913	0·876	0·678	0·505
0·250	0·958	0·920	0·886	0·698	0·527

[1] See footnote at end of § 71.

gives values of the efficiency of a screw pair, calculated from the formula already given, for a number of probable values of tan α (0 to 0·25) and for five different values of tan ϕ. The excessively low values of the efficiency for ordinary values of α, when the friction factor becomes large, is well shown in the table, and the vital importance of thoroughly good lubrication in any screw which is transmitting work needs no further emphasis.

The line of axial pressure, PN, does not always lie along one side of the screw parallel to its axis, but may be distributed over the whole thread equally, its resultant being coincident with the axis of the screw. But as in that case the effort (whether applied to the screw by a single lever or not), distributes itself round the thread in exactly the same way, no error is caused by summing each one up, as we have done, and treating it as if acting at one point only.

By the moment of the friction in a screw is generally meant the moment of the additional effort taken up in overcoming the friction, that is $MM_1 \times r$ in Fig. 337. The work done against friction in a given time is equal (as with a turning pair) to this moment multiplied by 2π and by the number of revolutions of the screw in the given time, or

$$2\pi r n. MM_1.$$

It will be noticed that MM_1 is not itself the real frictional resistance, which is represented by Q_1P, but is greater than it in the ratio $\frac{1}{\cos\alpha}$. In each revolution, however, the frictional resistance is overcome through a distance greater than $2\pi r$ in the same ratio (a distance equal to the length of one turn of the helix), and therefore the work done against friction is the same whether it be calculated from MM_1 or from Q_1P.

We have supposed the screw thread to be of square

section, so that the profile of the thread (as seen in the nut section of the figures) is at right angles to the direction of axial pressure. In a screw so made, other things being equal, the friction is a minimum. With a screw thread of triangular section, as Fig. 338, there is exactly the same addition to the frictional resistance as we have already noticed with a triangular guide block (p. 584). The normal pressure producing friction is increased from NP to NR (Fig. 338), *i.e.* in the ratio $\frac{1}{\cos a}$, as before. In consequence of this there is an inward pressure on the screw (and consequently an outward, bursting, pressure from the screw on the nut) of the magnitude RP, which may under some circumstances be very inconvenient.

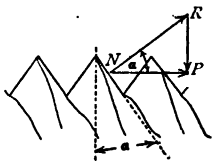


FIG. 338.

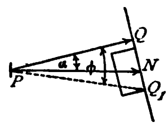


FIG. 339.

In practice the force PN does not act directly upon the screw thread, but generally upon the end, or upon some other portions of the worm spindle. At the surface where this pressure is transmitted there is therefore pivot friction, which in practice often causes a further very serious loss of effort, and therefore diminution of efficiency. This will be considered in § 75.

It is in many cases of great practical importance that a screw should not be able to run back, that is that no axial pressure, however great, should be able to turn it round. If the block in Fig. 339 be part of a screw thread, and PN the

axial pressure, then the ratio of driving effort to normal pressure must always be equal to $\frac{QN}{PQ} = \tan\alpha$. But the ratio of friction to normal pressure is $\frac{Q_1Q}{PQ} = \tan\phi$. So long therefore as $\tan\alpha$ is less than $\tan\phi$, that is so long as α is less than ϕ, the screw cannot run back,—no increase whatever of pressure in the direction PN can in any way move it.

§ 75.—FRICTION IN PIVOTS.

It has been pointed out in the last two sections that where there is any axial component of pressure in a turning or a screw pair, that pressure will cause friction separate from, and additional to, the friction proper to the pair. The rubbing surface may in this case be the faces of one or more collars on the shaft of the spindle, or may be formed by the end of the shaft itself. Generically all such surfaces, rotating in contact under *axial* pressure, may be included under the head of **pivots.**

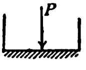


Fig. 340.

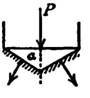


Fig. 341.

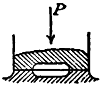


Fig. 342.

If P be the total pressure upon pivots such as those of Figs. 340 to 342, we may say at once that in the case of the flat-faced pivots the total frictional resistance will be fP, and in the case[1] of the coned pivot $f\frac{P}{\sin\alpha}$ if f be the friction-factor

[1] See § 72, p. 584.

suitable for the particular case. This information is, however, of no use to us unless we know also the mean radius at which we may assume the friction to act, which we have therefore to find. If we suppose that at first, as is intrinsically probable, the total pressure is uniformly distributed over the whole surface, then the mean radius of the friction must be two-thirds of the radius of the pivot (in the cases of Figs. 340 or 341, where the centre is not cut away). But with such a distribution of pressure *wear* will at once commence where the velocity of rubbing is greatest, that is at the outer diameter of the pivot, and will gradually extend itself inwards until the surfaces have so adjusted themselves (if this be possible) that the rate of wear is uniform over the whole. If the assumed friction-factor remained still uniform over the whole area, the wear at any point would be in direct proportion to the product of the velocity and the intensity of pressure at that point. If the wear is to be the same at every point, therefore, this product must have a constant value, so that the intensity of pressure at any point must vary inversely as its velocity, or simply as its radius. If, therefore, we suppose the whole surface to be divided into narrow concentric rings of equal breadth, the area of each ring will be proportional to its radius, and the intensity of pressure on each ring will be *inversely* proportional to its radius. The *amount* of pressure on each ring (*intensity* of pressure × area of ring) must, therefore, be the same. Under these conditions the mean radius of friction will be the half radius of the pivot, or (in the case of Fig. 342) the arithmetical mean between its inner and its outer radius. The moment of the frictional resistance would, therefore, be $f\frac{r}{2}P$ or $f\frac{r}{2}\frac{P}{\sin\alpha}$, and the work done per minute against friction

would be obtained by multiplying these quantities by 2π and by the number of revolutions of the pivot per minute, as $f \pi r P n$ or $f \pi r \frac{P}{\sin \alpha} n$ respectively.

This result is generally taken as correct, and probably enough it forms a reasonable working approximation, considering the very wide limits within which the factor f may vary. It involves, however, in the first place, the apparently impossible result that on some small, but not indefinitely small, area at the centre of the pivot, the intensity of pressure must be enormously great, so great as quite to destroy the metal locally. Of this we have no physical evidence in the condition of the surface of pivots after wear. But apart from this difficulty, the method of investigation involves two assumptions of which one is obviously wrong and the other doubtful. The first is the constancy of the friction-factor, and the second the possibility of uniform wear, or wear without alteration of the shape of the surface. As to the first, if we may apply here Mr. Tower's results (§ 71), we know that the friction-factor must vary at each point according to its velocity and pressure. If we suppose the whole surface of the pivot divided into equal small areas, the total frictional resistance on each would be about the same[1] for all areas having the same velocity, and otherwise would vary more or less as the square root of the velocity. The wear on each small area will be proportional to its velocity and its total frictional resistance, and therefore to $v\sqrt{v}$ or $v^{\frac{3}{2}}$. The velocity of each small area being proportional to its radius, we may, therefore, say that on these assumptions the wear at any point will be proportional to

[1] This supposes perfect lubrication, which is not unreasonable, but it applies results of journal friction to a pivot, which it is quite possible we are not entitled to do.

the square root of the cube of its radius, or $r^{\frac{3}{2}}$. But it is probable that at the extremely slow velocities existing close to the centre of a pivot the friction-factor must be proportionately higher than elsewhere, and the wear, therefore, greater than we have assumed. Further, the rapid wear at the periphery of the pivot must speedily reduce disproportionately the intensity of pressure there to such an extent that the constancy of frictional resistance on equal areas is no longer even approximately true. Starting from some simple assumptions, such as those first mentioned, as to the distribution of pressure and the value of the friction-factor, it is of course easy to calculate mathematically the conditions of wear, the best form of "anti-friction" pivot, and so forth. In fact, however, the actual physical conditions under which a pivot wears are only sufficiently known to show that the usual assumptions about them are entirely misleading. Until the physical side of the matter is more completely studied, it does not seem as if further purely mathematical investigation could of itself lead to any useful result.

So long as our knowledge of the probable mean friction-factor for a pivot is as uncertain as it is at present, the formulæ given on p. 598 for frictional moment, and work done against friction, no doubt give results sufficiently accurate for the rough approximation which is all we can at present hope to obtain. It seems almost certain, however, that the actual mean radius of the frictional resistances is *greater* than the half radius of the pivot, lying between it and the two-thirds radius.[1]

The total axial load on a shaft is often extremely great in proportion to its area, so great that without increase of area it would cause so great an intensity of pressure as to bring

[1] The efficiencies given in the table in footnote to § 71 include the frictional loss in the pivot as well as in the screw friction.

about inconvenient and probably irregular wear. To obviate this, "thrust collars" (Fig. 343) are often used (as in marine engines to take the thrust of the screw propeller), by multiplying which any required amount of surface can be obtained. In actual pivots the pressure is often distributed through several disks (Fig. 344) placed one below the other. If these be properly lubricated, each will move relatively to the one next it, and the sum of all the relative rotations will

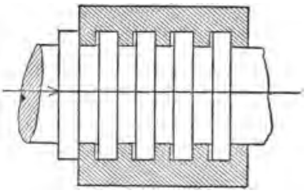


FIG. 343.

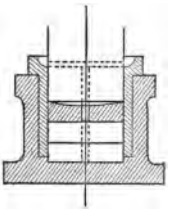


FIG. 344.

be equal to the whole rotation of the shaft relatively to its bearing, which is thus (more or less) uniformly distributed among the disks. By this means, although the pressure is not less than it would otherwise be, the velocity of rubbing, and therefore the wear of each pair of surfaces, is much reduced and rendered more uniform. Such bearings work very well in practice under very heavy pressures.

§ 76.—FRICTION IN TOOTHED GEARING.

IN toothed gearing of all kinds a very considerable amount of work is wasted in overcoming the resistance of the teeth to sliding on one another, as we have seen that they must do.[1] In § 18 we saw how to obtain the amount of sliding,

[1] §§ 18 and 71.

or distance through which rubbing takes place, during the whole contact of any pair of teeth. We saw further that the mean velocity with which the teeth slid upon one another might be expressed[1] as

$$= \frac{s}{ra} V = \frac{ra}{4}\left(\frac{1}{r} + \frac{1}{r_1}\right) V.$$

The first expression requires measurement of the tooth profiles (for s), the second does not. The value of ra, the distance moved through by a point on the pitch circle while a pair of teeth remain in contact, requires to be known in both cases. In order that two pairs of teeth may always be in contact ra must not be less than twice the pitch, although in practice it is not uncommonly only 1·6 to 1·8 times the pitch. If we insert $2p$ in the equations instead of ra we get

$$v = \frac{s}{2p} V = \frac{p}{2}\left(\frac{1}{r} + \frac{1}{r_1}\right) V.$$

The work lost *per second* (the velocities being supposed to be in feet per second) by friction between the teeth will be found by multiplying either of these expressions by the frictional resistance fP, where f is the friction-factor and P the mean total normal pressure between the teeth.

Work lost in friction per second

$$= \frac{s}{2p} V.fP = \frac{p}{2}\left(\frac{1}{r} + \frac{1}{r_1}\right) VfP.$$

The useful work done per second is (very approximately) VP,[2] so that the efficiency of the wheel gearing is

$$\frac{1}{1 + \frac{s}{2p}.f} = \frac{1}{1 + \frac{p}{2}\left(\frac{1}{r} + \frac{1}{r_1}\right)f}$$

and the counter-efficiency

[1] See p. 128. Note that r and r_1 are here written for r_1 and r_2.

[2] This assumes the pressure P to be in the direction of motion of the teeth instead of normal to their surfaces.

$$1 + \frac{s}{2p} \cdot f = 1 + \frac{p}{2}\left(\frac{1}{r} + \frac{1}{r_1}\right)f.$$

In either case $\frac{ra}{2}$ may be substituted for p when necessary. Surfaces such as those of wheel-teeth are often enough rough, and work with very imperfect lubrication, so that f is comparatively large; but in spite of this the actual loss by tooth friction is comparatively very small, much smaller than it is often imagined to be. Thus, for example, with f taken as much as 0·2, the efficiency of a pair of wheels of 2 and 6 feet diameter, with teeth of 3 inches pitch, is still about 97 per cent.,—so far, that is, as mere friction between the teeth is concerned.

It must be remembered that instead of $\left(\frac{1}{r} + \frac{1}{r_1}\right)$ in the above equations, $\left(\frac{1}{r} - \frac{1}{r_1}\right)$ must be used if one of the wheels (whose radius is r_1) is an annular wheel.

We have calculated above the *mean* value of the efficiency of transmission by toothed gearing. It is important now to see how that efficiency can be found graphically for any single position of the teeth. Let a and b (Fig. 345) be a pair of spur wheels turning about A and B respectively, and in contact at O. Let a_1 and b_1 be the circles with which the teeth have been described; further, let O_1 and O_2 be the first and last points of contact, respectively, of a pair of teeth. Let a be the driving wheel and MN the driving effort, while the resistance on b is assumed to act at C in the direction CF. Without friction we should find the resistance from the effort, for position of contact at O_1, as follows: Join O_1O,—this gives us the direction normal to the surfaces of the teeth, and therefore the direction of pressure between them. The wheel a is balanced under a known force MN, a

resistance acting along OO_1, and the sum of these two, which must pass through A. We find (as on p. 273) the join E of the directions MN and OO_1, and resolve the force MN in the directions EA and EO. This has been done in the triangle MNR, where RN is the pressure on the tooth surfaces. To find the resistance at C we have only to repeat a similar operation, resolving RN in the directions FC and FB. This gives us NM_1 for the resistance. In this case, of course, there being no friction, and both effort and resistance being assumed to act at the radius of the pitch circles, $MN = NM_1$. Taking now the same position of the mechanism, but assuming a friction-factor $= \tan \phi$ for the rubbing of the teeth, we can set off O_1E_1, making the angle ϕ with O_1E, the normal to the surfaces, and resolve MN parallel to E_1A and E_1O_1. This gives us SN for the sum of the pressure and friction at O_1. Carrying this on to b, and resolving SN in the directions F_1B and F_1C, we get for the net resistance NP instead of NM_1, the efficiency being $\frac{NP}{NM_1}$, which in this case is equal to $\frac{NP}{NM}$. The small arrows at O_1 show the direction in which the teeth slide on each other, which determines the position of O_1E_1. The relative motion of the teeth continues the same in sense (although its velocity diminishes) until O is reached. Here the point of contact of the teeth is also the virtual centre, and there is no sliding and therefore no friction,[1] so at this instant the efficiency of transmission is unity. It is in fact equal to the efficiency of transmission of two plain cylinders rolling on one another without slipping. At any intermediate point between O_1 and O the efficiency lies between $\frac{NP}{NM_1}$ and

[1] The resistance to rolling, which is sometimes called "rolling friction," is here disregarded, as being practically negligible in comparison with the friction proper.

unity, its value becoming greater as O is approached. After the centre is passed the efficiency again diminishes, but not so rapidly as before, because the sense of sliding of the teeth is now reversed (as shown by the arrows at O_2). The line O_2E_2, whose direction is that of the sum of the pressure and frictional resistance, is now *more* inclined to the line of centres than the normal O_2O, whereas before O was reached the

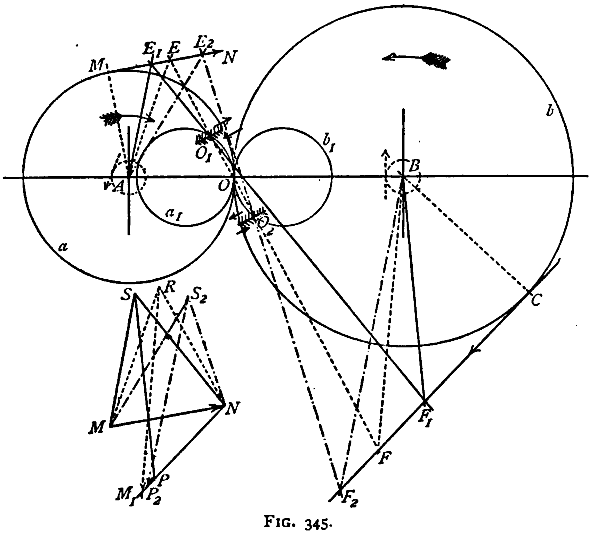


FIG. 345.

corresponding line (O_1E_1) was *less* inclined to the line of centres than the normal. This appears to be the real explanation of the statement so often made that the frictional resistance of the teeth as they approach the line of centres is greater than their frictional resistance as they recede from it. This is often expressed by saying that the friction during the "arc of approach" is greater than during the "arc of recess."

It will be noticed that during approach it is the *roots*[1] of the driving teeth which act upon the *points* of the driven teeth, while during recess the conditions are reversed, and the *points* of the driving teeth act on the *roots* of the driven. In order therefore to increase the efficiency as much as possible wheels have sometimes been made with only point-teeth upon the driver and root-teeth upon the follower. Such teeth have no contact before reaching the line of centres, and if they are to work well the arc of recess should therefore be made much greater than usual.

If E_1 be the efficiency of a pair of wheels at the commencement of contact of a pair of teeth, and E_2 at the close of the contact, then an approximation to the mean efficiency E, as close as is generally obtained from the formulæ on page 602 above, is given by

$$E = \frac{E_1 + E_2}{4} + 0{\cdot}5.$$

For most practical purposes $E = \frac{E_1}{2}$ is quite sufficiently accurate, and can be found of course by the very simplest construction.

If it is required at the same time to take into account the frictional resistance of the shafts, nothing more is necessary than the construction of Fig. 334 in § 73. Instead of drawing the lines from E, E_1, F_1, &c., through A and B, they must be drawn to touch the friction-circles which have these points as centres, the side on which they touch being determined as on p. 589. In Fig. 345 they touch to the left of the centres in both cases as dotted.

[1] See p. 126.

§ 77.—FRICTION IN LINKS AND MECHANISMS.

THE determination of the whole frictional resistances, or of the total efficiency of transmission in a link or in an entire mechanism, involves no more than the right use and combination of the constructions already given, which we shall now illustrate by some examples. Let it be borne in mind, before proceeding, that what we are finding here is only the efficiency of a link or mechanism *in one particular position* under the action of the given forces. Its efficiency varies as its position changes, and the value of its *mean* efficiency throughout one revolution, or other complete cycle of changes of position, requires to be found separately. This matter will be considered later on.[1]

We have seen how to determine the resultant direction of friction and surface pressure in a pin joint or turning pair. The most important point now before us is the corresponding determination in the case of a *link* connected with its neighbours by two such pairs, such for example as an ordinary coupling or connecting rod. The direction of the resultant just mentioned may in this case either cross the axis of the rod or lie parallel to it, and this resultant has in general four possible positions. Its direction line may conveniently be called the **friction axis** of the rod, and is always different from its geometrical axis. The four cases just mentioned are shown in Figs. 346 to 349, of which we shall first look at Fig. 346 alone. The link b is the one of which we require to find the friction axis. The mechanism turns in the direction of the arrows on a and c, and c is the driving link. The forces f_1 and f_2 acting on b from c and a,

[1] Keep in view always, in working out the efficiency of a machine, the remarks at the end of § 71.

without friction, would have the sense of the arrows shown, and would coincide in direction with the axis of the link. The angle *bc* is (in the position shown) increasing, and the angle *ba* is simultaneously decreasing. The rotation of *b* relatively to *c* and *a* is, therefore, represented by the small arrows on *b*, contra-clock-wise or left-handed in both cases. The direction line of the sum of the force f_1 and the friction at its joint must touch the friction circle of the joint, and touch

FIG. 346.

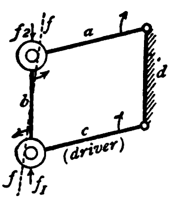
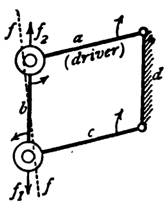


FIG. 347.



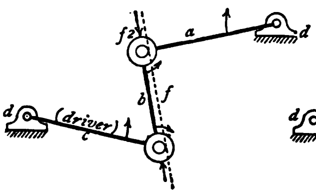


FIG. 348.

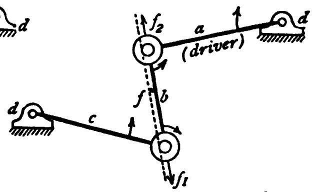


FIG. 349.

it on such a side as to *oppose* the rotation of *b* relatively to *c*, *i.e.* to oppose the motion of the pin in the eye, or of the eye over the pin. This direction line must, therefore, lie to the left of f_1, and by exactly similar reasoning we can see that it must lie to the right of f_2. But the directions of the reactions at the two ends of the link must coincide;—if they did not, there would be an unbalanced moment acting upon *b*, and the mechanism could not be in equilibrium. Hence the

friction axis f can be drawn at once as a line touching the two friction circles, the one to the left and the other to the right of its centre.

If the motion of the mechanism had been the same, but with the link *a* the driving link instead of *c*, we should have had the case of Fig. 347. The sense of f_1 and f_2, as forces acting upon *b*, would have been reversed. The rotation of *b* relatively to *c* and *a* would, however, have remained unchanged. The friction axis f would, therefore, have crossed the axis of the link in the reversed sense to that of the last case. Its position is shown in the figure.

In the mechanism sketched the link *b* has the same sense of rotation relatively to *a* and to *c*. But in such a mechanism as that of Fig. 348, it has opposite senses of rotation relatively to its adjacent links. The angle *bc* is increasing, and the angle *ba* decreasing. The sense of rotation of *b* relatively to *c* is right-handed, and relatively to *a* left-handed. The link *c* is the driving link, and the friction axis f touches both friction circles on the *same* side, and lies parallel to the geometrical axis of the link. In Fig. 349, the link *a* is taken as the driving link, everything else remaining unchanged. The change in the friction axis exactly corresponds to that in Fig. 347 above. It remains parallel to the axis of the link, but lies on the opposite side of it.

In dealing with a link in this way it is essential to remember that the sense of the forces must always be taken as that corresponding to their action *on* the link, and not *from* it.[1] Similarly the sense of rotation opposed by the friction is that of the link itself relatively to its neighbour in each case, and not of its neighbour relatively to it. If these things are clearly kept in mind in working from link to link through

[1] See p. 589, § 73.

a mechanism, the necessary constructions will not give any trouble.

Before going on to any more complex cases we shall work out completely two examples from those we have just looked at, taking first the mechanism of Fig. 350. A force f_c ($=RS$) acts at C; we require to find its balance at A, taking into account friction at all the four pins. The circles at the joints represent the friction circles, not the pins, which are omitted for the sake of clearness.[1] The given force f_c intersects the

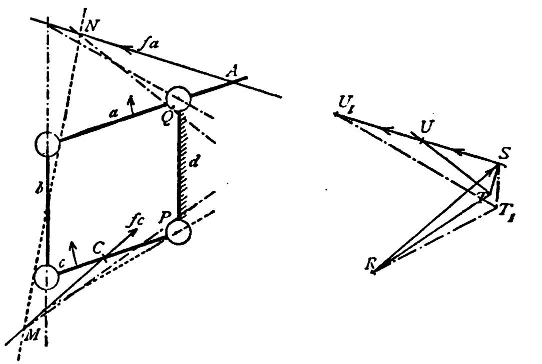


FIG. 350.

friction axis of b in M. We first resolve it in the direction of that axis, and along a direction through M touching the friction circle of cd at P. The sense of the component in the last-named direction is from P to M, which determines the side on which it shall touch the friction circle. This resolution gives us ST (in the figure separately drawn) as the component of f_c acting along the friction axis of b. The required force f_a cuts the friction axis in N, and to find it we

[1] As to size of the friction circles see p. 591.

have only further to resolve ST (reversed in sense, as acting on a and not on c) in the directions NA and NQ, the last being determined in the same way as MP. This gives us the triangle SUT, of which the side SU represents the required value of f_a. The dotted lines give the corresponding construction disregarding friction, so that $\frac{SU}{SU_1}$ is the total efficiency of the mechanism for the position sketched.

Exactly the same problem is solved in Fig. 351 for the mechanism of Fig. 348. The same lettering is used, so that the force lines do not need to be traced out in detail. In both cases $\frac{SU}{SU_1}$ is the total efficiency of the chain. In both

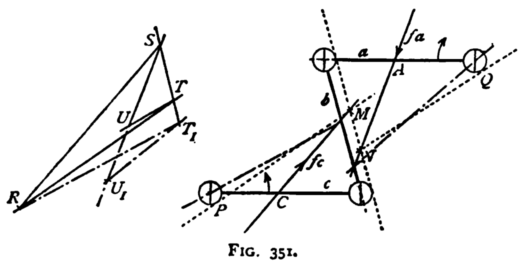
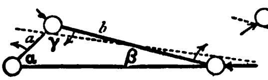


FIG. 351.

cases, it will be seen, the construction is practically identical with that of Fig. 127, § 40, with the substitution of P and Q for the two virtual centres, and of the friction axis of the middle link for its geometrical axis. The more general construction of Fig. 128, § 40, cannot be applied here with any approach to accuracy.

The position of the friction axis of the connecting rod of an ordinary steam-engine undergoes all the four changes of Figs. 352 to 355, during each revolution. From the commencement of the forward stroke the angle β increases and

the angle γ diminishes, and the friction axis has the position shown in Fig. 352, where the small arrows on b show its sense of rotation relatively to the crosshead and the crank, its two adjacent links. When a becomes 90°, *i.e.* when the crank is in its mid-position, the angle β has obtained its maximum value, and while the crank is in its next quadrant it continually diminishes, the angle γ still diminishing also. The position of the friction axis is shown in Fig. 353. During the next quadrant of the crank's motion, the angles β and γ both increase, the forces acting on the rod change sign (the engine now making a backward stroke), and the friction



352.

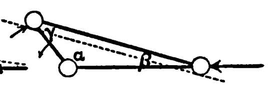


FIG. 353.

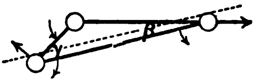


FIG. 354.

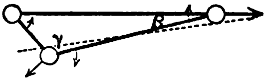


FIG. 355.

axis simply changes sides, remaining parallel to the axis of the rod (Fig. 354). In the last quadrant, γ still increases, but β diminishes. The friction axis takes the position shown in Fig. 355, which is just reversed (corresponding to the reversal of the force signs) from that of Fig. 352.

Had the machine been a pump instead of an engine, so that the crank was the driving link instead of the piston, we should have had each position of the axis reversed. Thus Fig. 356 corresponds to Fig. 352. The forces have the same sense, but the sense of rotation of the crank, now the driving

link, is reversed.[1] The angle β is therefore diminishing and γ increasing, and the position of the friction axis is the same as formerly in Fig. 355, that is, reversed from its former position.

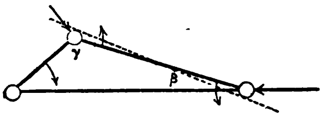


FIG. 356.

As an illustration of the determination of the efficiency of a mechanism containing both sliding and turning pairs, we cannot do better than take the ordinary steam-engine mechanism, such as is sketched in Fig. 357. The direction

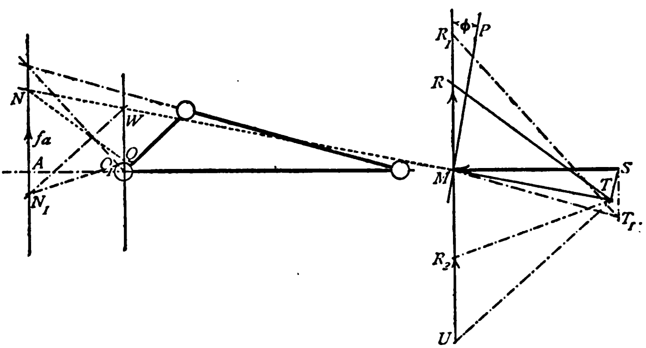


FIG. 357.

of the effort in an engine is always fixed, but the direction of the resistance varies very much, and upon its position in any given case the actual efficiency must depend. In the case

[1] If the sense of the crank's rotation were left unchanged, the sense of the forces would have to be reversed, and the result would be the same.

sketched the direction of the resistance is made such as it would be if the engine were driving machinery by means of a spur wheel or pinion of a radius equal to that of the point A. The circles represent, as before, the friction circles, and not the pins. We draw first the line MP, making the angle ϕ with the normal MR, and resolve SM, the piston pressure, in the direction of MP and of MN, the friction axis of the connecting rod. This gives us TM as the force acting in the latter direction. Changing its sign, and resolving it in the directions f_a and NQ, we find at once MR as the required value of f_a, the resistance at A. Without friction the resistance balanced would be MR_1; the efficiency is therefore $\frac{MR}{MR_1}$. If the friction circles have been enlarged in any ratio, on Professor Smith's plan (see p. 591), it must not be forgotten that the value of tan ϕ (the friction angle for the sliding block) must be enlarged in the same ratio. If this is not convenient, the efficiencies of the sliding pair and of the connecting rod and crank shaft must be determined separately, and afterwards multiplied together.

The loss of efficiency found by such a construction as this refers only, of course, to the particular forces which have been taken into account. There is no difficulty in finding similarly the losses caused by friction due to the weight of the moving parts. If in such a case as that of the last figure, for example, there be some very large weight, as that of a fly-wheel, upon the bearing, the frictional resistance caused by it may be estimated and allowed for separately. This is probably the most convenient plan, because any such resistance is the same for every position of the mechanism, so that one calculation serves for all. But it can also be found graphically with the greatest ease. Thus let MU (in the last figure) be any such weight, acting vertically down-

wards through the centre of the shaft. Adding MU to TM we get TU as the sum of the weight and the connecting rod pressure on the crank, and this sum must pass through the point W, the join of the lines of action of the weight with the friction axis. Through W draw WN_1 parallel to UT, and resolve UT in the direction of f_a and of N_1Q_1, which gives UR_2 as the new value of the force balanced at A, allowing for the journal friction caused by the weight MU.

The donkey-pump mechanism of Fig. 184 affords us a very instructive, but somewhat more complex, example. Let

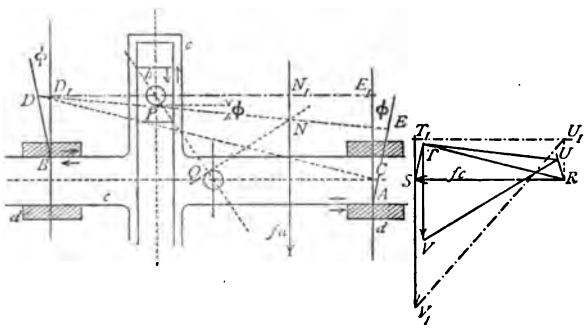


FIG. 358.

it be assumed that this mechanism is to be used for a steam-engine (where the main resistance is, as usual, to the rotation of the shaft) and not for a pump, and find its efficiency in the position shown in Fig. 358. Here we start with the given piston pressure $f_c = RS$, acting upon the sliding frame c. This is balanced by a pressure or resistance from the block to the frame, and also by side pressures in the two guides. We have in the first instance to assume the points A and B at which the resultant pressures in the guides act; these we have no means of determining. The pressure from the

guide to the link c will be upwards at A and downwards at B, and as we know the sense of motion of c relatively to the guides (as shown by small arrows) we can draw the lines AE and DB, making the friction angle ϕ with the normals AE_1 and D_1B, at A and B respectively. Without friction the direction of pressure from b to c would be simply the normal D_1E_1, which line is the *real geometrical axis* of the link b, which we know to represent (see p. 399) an infinitely long connecting rod. *With* friction the direction of pressure must be along the friction axis of b. Of this line we know, firstly, that it must touch the friction circle of the crank pin on the under side in the figure, as at P, for exactly the same reason as in Fig. 353 above, which represented the similar position of the slider crank mechanism. Secondly, we know that it must be inclined at an angle $=\phi$ to the normal to the surfaces of the block and frame. We can therefore at once draw it as PN or DE. We have now the condition that the link c is in equilibrium under four forces, of which one, f_c, is given completely, and the other three are given in direction only, as AE, ED, and BD. We can employ for resolution the construction used formerly in p. 312, § 41. Calling the pressures at A, B, and P, f_a, f_b, and f_p, respectively, we know that

$$f_c + f_a = -(f_b + f_p).$$

But the sum of f_c and f_a must pass through C, the join of their directions, and similarly the sum of f_b and f_p must pass through D. To find f_p then we have first to resolve f_c along AC and DC, which gives us TR (see separate figure) for the component along DC, which is the sum of f_b and f_p. Next we resolve this component in the directions DN and DB, which gives us TU as the required reaction between b and c. Had there been no friction we should have had to resolve f_c along the direction AE_1, E_1D_1, and D_1B, and the

force polygon would have been the simple rectangle RST_1U_1 (see also p. 311). To complete our problem, let f_a be the given direction of the resistance upon the crank shaft, whose magnitude we require to find. We find N, its point of intersection with the friction axis of b, just as in Fig. 357, and resolve TU along the direction of f_a and of NQ, the point Q being found as before. This gives us TV for the resistance which we had to determine. Without friction we should have had to resolve T_1U_1 in the directions of f_a and of N_1O, which gives us T_1V_1. The efficiency of the mechanism as a whole is, therefore, for this particular position,

$$\frac{TV}{T_1V_1}.$$

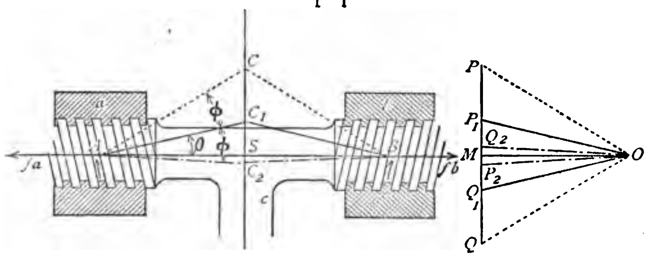


FIG. 359.

It will be sufficient to take one example with a screw; for instance the right- and left-handed screw coupling of Fig. 359. The coupling is pulled with a constant tension $f_a=f_b$, it is required to find the moment necessary to turn the screw in either direction under this tension. Suppose first that the screw has to be turned so as to tighten up the coupling. Through A and B draw normals to the screw threads, meeting at C_1. Set off $MO=f_a=f_b$. Through O draw $OP_1 \parallel BC_1$ and $OQ_1 \parallel AC_1$, making the direction P_1Q_1 normal to the axis of the screw, that is in the direction of the intended

turning effort. Then *without* friction the effort P_1Q_1 applied at a radius equal to the mean radius of the screw thread, will be just sufficient to turn it. The moment of the effort will be $P_1Q_1 \times r$, if r be the mean radius of the screw. The sense of motion of the screw relatively to the nuts is shown by the small arrows at A and B. With friction therefore the directions C_1A and C_1B will be changed to CA and CB, the angles CAC_1 and CBC_1 being each $=\phi$. Drawing parallels to these lines in the force polygon we get PQ as the effort required instead of P_1Q_1, and the moment necessary to turn the screw, including frictional resistance, is $PQ \times r$, if r be, as before, the mean radius of the screw thread.

If the screw has to be slackened instead of being tightened up, the friction angle has to be set off in the opposite sense, as C_2A and C_2B. The corresponding effort, greatly less than before, and of course reversed in sense, is shown at P_2Q_2, the turning moment being $P_2Q_2 \times r$. We have already (§ 74) noticed the necessary relations between the magnitude of the angles ϕ and θ in order that the screw may not "run down." Here it will be seen at once that if $\phi=\theta$, C_2 would coincide with S, and the points P_2 and Q_2 in the force polygon would come together, so that the effort required to slacken the screw would be zero. If θ were greater than ϕ, C_2 would fall above S, and some effort would be required *to prevent the screw slackening itself.* As this would entirely destroy the usefulness of the coupling, the case is one in which a finely pitched thread (*i.e.* a small angle θ) is essential, and a too small friction-factor ($=\tan\ \phi$) detrimental instead of desirable.

We have already pointed out that the constructions of this section have for their object the determination of the efficiency of a mechanism in one particular position only. In general what is practically required is the *average*

efficiency of the mechanism during one complete cycle of changes of position, as, for instance, the average efficiency of the mechanism of a steam-engine during one complete revolution of the crank shaft. To obtain the efficiency of the mechanism in a number of different positions, and then find the average value of the efficiencies so obtained, would be a long process, because each determination of efficiency requires two complete constructions, one to determine the balanced resistance *with* and the other *without* friction, the efficiency being the ratio between these two quantities. But this is unnecessary. In every case we do, or easily can, start with a diagram of work, that is a curve (as Fig. 146, p. 321) whose ordinates represent pressures (here efforts), and whose abscissæ represent the distances through which these pressures are exerted. All that is necessary to do is to determine by construction the resistances *with* friction, and plot these out into a diagram whose base represents the distance travelled by the point at which the resistance acts (as A in Fig. 357 above). Apart from work done against friction this diagram would have an area equal to that of the effort diagram, as we have seen in § 43. Having, however, taken friction into account, its area will represent the *net* work done against useful resistance, and will be less than the area of the effort diagram by an amount corresponding exactly to the work expended in overcoming frictional resistances. The ratio between the areas of the two diagrams will be the mean efficiency of the whole mechanism.

It is not necessary that we should give here any detailed example of this determination for a whole mechanism; all the necessary constructions for it have been given very fully. It will be sufficient to give the simple case of the determination of the work lost in the friction of the guide block of a steam-engine, and the average efficiency of the guides.

Fig. 360 represents this case (which we have already looked at) for one position, Fig. 361 shows the determination of the average efficiency. AB (Fig. 360) is the known piston effort, balanced at B by a resistance in the direction of the friction axis of the connecting rod, BM, and by a normal pressure and frictional resistance whose sum lies in the direction BB_1, making an angle ϕ to the vertical. Drawing the force polygon we get BM for the pressure transmitted through the connecting rod, and BA_1 for the effort which would be required to balance this pressure if there were no friction. The efficiency is therefore, in this position, $\frac{BA_1}{BA}$

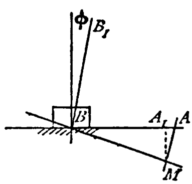


FIG. 360.

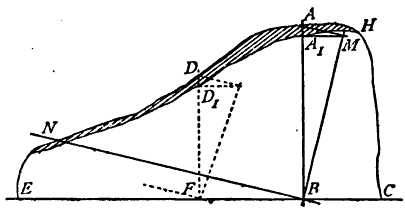


FIG. 361.

If an effort diagram has been drawn, BA will be equal to its ordinate at the point B, which must have been turned down into its present position, and we must now turn up BA_1 above B as an ordinate for our new (net effort) curve. But this turning down and up of lines is somewhat inconvenient, and it is much more handy to turn the whole construction through a right angle, and work it as in Fig. 361. Here $CADE$ is the effort diagram (an indicator card drawn to a straight base), BA the effort at B, and NB the direction (as BM in Fig. 360) of the friction axis of the connecting

rod. Through B draw BM at right angles to BN, and through A draw AM making an angle equal to ϕ with the horizontal. Then project M to A_1 on the line BA_1, and the required point on the net effort curve is at once obtained. It will be seen at once that the figure BA_1AM in Fig. 361 is identically equal to the similarly lettered figure in Fig. 360. Each of its sides is, however, turned through 90° to suit the direction in which the effort has been originally set out. A similar construction for FD gives us FD_1 for the net effort at D. The shaded area $HADED_1A_1$ represents the work expended in overcoming the friction of the guide block, and the ratio of areas $\frac{CA_1D_1E}{CADE}$ gives the mean efficiency for the motion from C to E.

The plan just used of turning the construction through a right angle is one which the student will find useful in a number of cases, especially where work or energy diagrams are concerned, but it is not necessary to give further examples of it.

In finding the mean efficiency of any mechanical combination other than the very simplest, it is very desirable that for at least one position the student should make the complete determination of resistance both with and without friction, and should see that at each separate joint there is a loss of efficiency. Without this double determination there exist no ready means of checking possible mistakes in the position of friction axes, &c., which may notably affect the resultant efficiency, without, however, making it so conspicuously wrong as to be otherwise evident.

In a very large number of machines, more or less complex in appearance, the friction (so far as it is caused by known and measurable forces) can be quite easily estimated by the methods given by treating them as a sequence of separate

mechanisms, and graphically or otherwise combining the whole. In really complex mechanisms, however, such as those of Figs. 128 and 240, the determination of the frictional efficiency is much more difficult. Our knowledge, however, of the real value of the friction-factor in any particular case is so very vague that the error of an approximation which is mathematically exceedingly rough may still be much less than the probable error of our estimation of this most essential element in our data.

An excellent collection of examples of graphic frictional estimations will be found in the *Zur graphischen Statik der Maschinengetriebe*[1] of Professor Gustav Hermann, of Aachen. These include a stone-breaking machine, pulley tackle, screw-jack with worm gearing, geared crane, and various forms of steam-engine.

§ 78.—FRICTION IN BELT-GEARING.

It was mentioned in § 71 that there is a large and important class of cases in which it is desired that the surfaces between which friction occurs shall *not* move relatively to each other; and in which the frictional resistance alone is relied upon to prevent this motion. In these cases no work is expended in overcoming friction, because no motion takes place under it; the frictional contact between the surfaces does not affect the efficiency of the apparatus; and instead of wishing to diminish the frictional resistance as much as possible, it is essential to the working of the machine that it should have at least some definite, and generally very large, amount.

[1] Brunswick, Vieweg u. Sohn, 1879.

The most important case of this kind is that of the transmission of work by means of belt-gearing. Let there be given any pulley, as in Fig. 362, with a strap resting upon it, and at each end of the strap a weight. For mere static equilibrium, so long as the pulley works frictionless in its bearings, $W_2 = W_1$, and the smallest addition to either

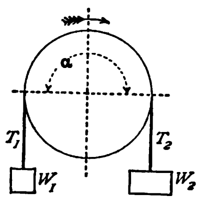


FIG. 362.

weight will cause it to descend, lifting the other. But suppose the pulley to be fixed, so that its rotation is entirely prevented, and that it be desired to make W_2 large enough to lift W_1. In order that W_2 may move, it has now not only to balance W_1, but also to overcome the whole friction between the strap and the pulley caused by the tensions T_1 and T_2 in the strap. If we call this whole frictional resistance F, the condition of possible motion is

$$W_2 = W_1 + F, \text{ or } T_2 = T_1 + F, \text{ and}$$
$$W_2 - W_1 = T_2 - T_1 = F.$$

The value of F depends essentially upon (*a*) the friction-factor for the belt and pulley surfaces, (*b*) the tension in the belt, and (*c*) the angle of contact α. It can be shown by integration [1] that for flat pulleys

$$T_2 = T_1 \epsilon^{\pm f\alpha}$$
$$F = T_2 - T_1 = (\epsilon^{\pm f\alpha} - 1);$$

[1] The proof is therefore not given. It will be found in all books on the subject which utilise higher mathematics, *e.g.* Rankine, *Machinery*

while if the pulley be grooved with a V-shaped groove of angle 2θ, the power $\frac{f\alpha}{\sin\theta}$ must be substituted for $f\alpha$. In these expressions ϵ is the number 2·72 (nearly), the base of the natural system of logarithms, f is the friction-factor, α is the arc of contact *in circular measure*. In the case sketched in the figure, where motion is to take place in the direction of T_2, T_2 is greater than T_1, and the index in the formula must be used with the positive sign. If it had been required to find to what amount T_2 would have to be reduced before W_1 could begin to move downwards against it, the same formula would be used, but with a negative instead of a positive index. Thus if $W_1 = 100$ lbs., $\alpha = 180° = \pi$, and $f = 0·4$, the smallest value of T_2 which will lift W_1 will be $100\ (2·72^{0·4\times\pi}) = 351$ pounds, and the value to which T_2 must be reduced in order to allow W_1 to fall must be $100\ (2·72^{-0·4\times\pi}) = 28·5$ pounds. Or otherwise, if $T_1 = 500$ pounds, and $T_2 = 1$ pound, f remaining as before, we can find the value of α, in order that the system may be balanced,[1] as

$$\alpha = \frac{1}{f}\log_\epsilon\frac{T_1}{T_2} = 15·54.$$

This is in circular measure, and is therefore equivalent to $\frac{15·54}{2\,.\,\pi} = 2·47$ complete turns. Thus with the assumed (very large) friction-factor, two and a half turns of a cord round a fixed pulley would give friction enough to enable 1 pound to

and Millwork, art. 310 A, or Cotterill, *Applied Mechanics*, art. 123. Professor Cotterill also gives a graphic construction for finding the variation of tension and frictional resistance along the belt.

[1] It will be remembered that the natural logarithm, or logarithm to the base ϵ, of any number can be obtained by multiplying its common logarithm by the number 2·30.

hold against 500, or in other words to prevent a weight of 500 pounds from running down. The hauling of a rope round a capstan is of course a familiar example of this.

We have now to apply these results to movable pulleys, such as those used in belt-gearing. Let us suppose we have a belt pulley such as is shown in Fig. 363, where the resistance to the motion of the pulley is a weight W, acting at an arm r, the radius of the pulley being R. In the first instance suppose the weight to rest on the ground, and that T_2 ($= T_1 e^{fa}$) is very small. At first the strap will slip round on the pulley and the weight will remain unmoved. But if we continuously increase T_2 we not only increase T_1, but increase

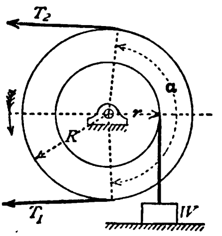


FIG. 363.

also the frictional resistance to slipping, $T_2 - T_1$. At some point the moment of the frictional resistance becomes equal to the moment of the weight, *i.e.*,

$$(T_2 - T_1)R = Wr,$$

and after this the pulley turns, lifting the weight, and the strap ceases to slip on the pulley. This is the condition under which all belt pulleys work. With any further increase of tension in the strap we do not increase $(T_2 - T_1)$, so long as the motion is uniform, as the equality of moments just

stated must always exist.[1] By substitution in our former equations we now have

$$T_1 = W\frac{r}{R}\left(\frac{1}{\epsilon^{fa} - 1}\right);$$

$$T_2 = W\frac{r}{R}\left(\frac{\epsilon^{fa}}{\epsilon^{fa} - 1}\right);$$

or if we take $r = R$ and write n for ϵ^{fa}

$$T_1 = W\frac{1}{n-1};\ T_2 = W\frac{n}{n-1}.$$

If the pulleys are of different sizes the value of a should be taken on the smaller. The *mean tension* in the belt is $\frac{T_2 + T_1}{2} = \frac{W}{2} \cdot \frac{(n+1)}{(n-1)}$. This is the mean tension necessary to drive a pulley the resistance to whose motion is equivalent to W pounds at its own peripheral radius, with the given values of f and a.

Rankine has pointed out that if f be taken equal to 0·22 (probably too low a value, and therefore on the safe side) and a be taken $= \pi$ $(= 180°)$, and $r = R$, we get $\epsilon^{fa} = 2$ very nearly, and our equations simplify to

$$T_1 = W;\ T_2 = 2W;\ T = \tfrac{3}{2}W.$$

These approximations are practically quite useful, for here as in the other cases we have been dealing with, we have nothing even approaching to exact knowledge of the value

[1] There is of course nothing to prevent our still increasing the tensions, say by pushing the pulleys further apart by screws, while at the same time the motion is kept uniform. As in this case $(T_2 - T_1)$ cannot increase, we must suppose that such additional tension is equally divided between the two sides of the belt. The tensions only stand to each other in the ratio given by the formulæ so long as they do not exceed the amounts just necessary to balance the given resistance under the given conditions.

of the friction-factor (which in extreme cases may vary from 0·15 to 0·55) while at the same time there may be some considerable, but unknown, excess of tension in the belts, owing to the cause mentioned in the footnote on the last page.

In modern practice there are many cases in which, instead of flat pulleys with broad flat leather belts, grooved pulleys are used, with hempen rope belts. In this case the gripping of the rope in the groove causes an increase in the frictional resistance, and enables a given resistance to be overcome with a smaller tension. Thus if the angle of the groove be 60°, and $f = 0{\cdot}22$ as before, $\epsilon^{\frac{fa}{\sin 30^\circ}} = 4$ nearly, and by substitution

$$T_1 = \frac{W}{3};\ T_2 = \frac{4W}{3};\ T = \frac{5W}{6}.$$

With wire rope gearing it has been found injurious to the rope to allow it to bite against the sides of the groove, and the grooved form of pulley is used merely to prevent the rope slipping off; the wire rope rests on a flat surface of wood, leather, or other special material (with which it has a high friction-factor) at the bottom of the groove.[1]

Professor Rankine[2] was the first to point out the influence of *centrifugal tension* in a belt, which although generally quite negligible, may become very important when the velocity is high. If w be the weight of a belt in pounds per foot run, v its velocity in feet per second, then the centrifugal tension in the belt will be $\frac{wv^2}{g}$ pounds, which in certain cases forms an addition to T_2 which cannot be left out of account.

[1] Chaps. x. and xi. of the third edition of Professor Unwin's *Elements of Machine Design* contain an excellent account of recent practice in these matters, as well as theoretical investigations relating to them.

[2] *Machinery and Millwork*, art. 381, &c.

If, as often happens, we do not know the resistance W, but require to find the tensions in a belt which shall transmit a given horse-power P, we can easily find them. For

$$P = \frac{W \cdot 2\pi R \cdot t}{33000} = \cdot 00019\, RWt$$

where t stands for the number of revolutions made per minute by the shaft. Then

$$W = 5250 \frac{P}{Rt};$$

$$T_1 = 5250 \frac{P}{Rt}\left(\frac{1}{n - 1}\right); \quad T_2 = 5250 \frac{P}{Rt}\left(\frac{n}{n - 1}\right).$$

The frictional resistance of a pulley shaft in its bearing is not in any way different from that of any other turning pair, but it is shown separately in Fig. 364 on account of its importance. There are, acting on the pulley, the two strap tensions T_1 and T_2, and the resistance, whose direction only (r) is

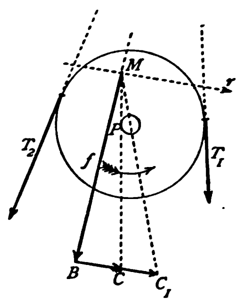


FIG. 364.

known. The sum of these three, plus the frictional resistance, must have a direction such as to touch the friction circle. We first add together T_1 and T_2 by any of the usual constructions. Their sum is represented in magnitude and

position by $MB = f$. The join of f and r is M. The resultant pressure has the sense from P to M, and must therefore touch the friction circle on the left side of the centre. We can at once resolve MB in the directions of r and of MP, which gives us BC for the resistance, allowing for friction. Without friction BC_1 is the resistance. The efficiency is therefore $\dfrac{BC}{BC_1}$.

In leather belting, and indeed in all such gearing, a small loss of efficiency, not included under any head yet mentioned, is caused by the work expended in first bending the belt or rope on to the pulley, and then straightening it again. Where the thickness of the belt (or diameter of the rope) is small in comparison to the radius of the pulley, this loss is very small, but where the diameter of the rope is proportionately large in comparison with the pulley, it may be very considerable. It will be further considered in § 80.

§ 79.—FRICTION BRAKES AND DYNAMOMETERS.

If we have a pulley revolving within and slipping upon a belt or strap in which certain known tensions exist, we are able without any trouble to find exactly the amount of work which is being done upon the pulley. We get in this way the very simple form of friction brake or dynamometer sketched in Fig. 365. The pulley revolves in the direction of the arrow. The tension T_2 is determined and kept constant by the weight W. The tension T_1 is known at each instant by means of the spring balance S. Quite independently therefore of any calculation, or any assumed value of the friction-factor, we can find the value of $(T_2 - T_1)$, and the work done on the pulley per minute will be

$$(T_2 - T_1)\ 2\pi Rt.$$

If the pulley be upon the shaft of an engine by which it is driven, we have in the above expression the value of the *net* work done by the engine, and the ratio of this work to the work shown by the indicators, that is the work done in the cylinders, is called the **mechanical efficiency** of the engine. A brake of this sort, which can only measure work which it absorbs, and cannot transmit this work to other apparatus, is called an **absorption dynamometer.**[1] The belt used in this case is generally of leather, but it will be found often better to use a number of wires, of $\frac{1}{8}$ to $\frac{1}{4}$ of an inch diameter, making the pulley cylindrical instead of barrelled. There is no necessity for limiting the arc of contact

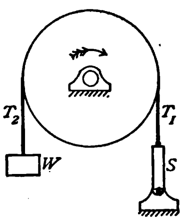


FIG. 365.

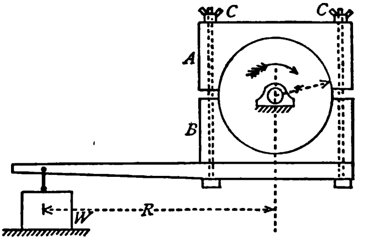


FIG. 366.

to 180°. By making it one whole turn, or even more, the value of the variable tension T_1 may be much reduced, and errors caused by its variations made correspondingly smaller.

The most common form of friction brake, that known generally as **Prony's** dynamometer, is that shown in Fig. 366. Here a pulley is enclosed in two blocks A and B, which can be tightened together by screws C, and to one

[1] A brake which does not absorb, but transmits, the work which it at the same time measures, either keeping the rate of work constant or measuring its variations, is called a **transmission dynamometer.** See the references cited at end of this section, and also Hirn's *Les Pandynamomètres* (Gauthier-Villars, 1876).

of them is attached an arm from which hangs a known weight W at a known radius R. The radius of the pulley is r, the frictional resistance we may call F. Suppose the pulley set in motion, the weight W resting on the ground and the screws C slack. The pulley simply revolves inside the blocks. The work done on the pulley corresponds to the (unknown) moment of friction Fr. If now the screws be gradually tightened up, the value of F gradually increases, until at length the moment of friction Fr is equal to the fixed moment of the weight WR. After this, if the conditions remain constant, the pulley rotates, keeping the lever floating and the weight W just off the ground. We have then continually

$$WR = Fr;$$

and the work done by the engine, which is

$$2\pi rFt \text{ foot-pounds per minute,}$$

can be expressed in known quantities as

$$2\pi RWt \text{ foot-pounds per minute,}$$

(t being as before the number of revolutions per minute). The work done is thus exactly equivalent to the continual winding up of a weight W, out of an infinitely deep well, on a drum of R feet radius.

The Prony brake is, as in the last case, an absorption dynamometer. At the Düsseldorf engine trials in 1880, the lever was placed on the opposite side of the brake blocks, and the corresponding upward reaction was furnished by the table of an ordinary platform weighing-machine, on which it was allowed to rest. This is a very convenient and practically accurate way of using the brake, and deserves to be more widely known than it has been.

Such a brake as that sketched, although with careful

treatment it will give reasonably accurate results, is of course a somewhat rough instrument. The weight W will not remain perfectly steady, but will rise and fall, and the radius R does not remain perfectly constant. Moreover, such changes in the friction as continually occur with the varying degrees of wetness of the brake-surface cause irregularities which can only be kept within limits by constant attention to and alteration of the tightness of the screws. For accurate experiments a brake should, as far as possible, compensate for all such changes automatically, and to a considerable extent this is the case with the arrangement shown in Fig. 367.[1] Here a

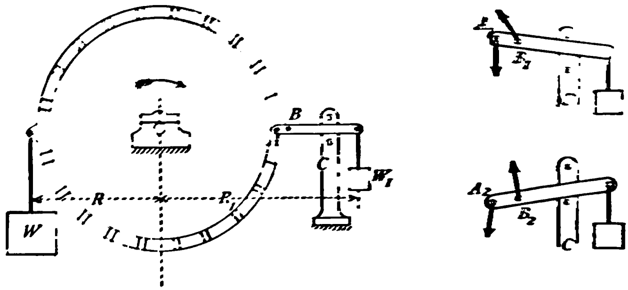


FIG. 367.

ring of wooden blocks connected by hoop-iron straps takes the place of the large wooden blocks, and the weight W hangs from a pin attached to the straps. Its moment is WR, as before. On the opposite side the straps are cut, and the two ends attached at different points AB to a small lever,

[1] This is the arrangement used in the experimental engine at University College. In the original form of the "pendulum" lever, (which is the invention of Mr. Appold), the form most generally used, the pins AB are placed at the bottom of the brake, and the lever points upwards, and is allowed to press against a side block with an unknown pressure. This may in cases cause a very sensible error in the estimation of work done, an error always in excess.

from which hangs a small adjustable weight W_1. This lever can vibrate, if the brake oscillates, between two stop-pins on a standard *C*. The radius of W_1 is R_1. While the brake is working steadily in the position shown, lever, strap, and weights are virtually all one piece, and the equivalent moment is $WR - W_1R_1$. If the friction diminishes, the weight *W* falls, the lever rises to the upper stop, and takes the position A_1B_1, thereby tightening itself up. If, on the other hand, the friction increases, the weight lifts, and the lever falls against the lower stop and takes the position A_2B_2, thereby slackening the strap again. By making the play between the stops very small, the brake will keep itself steady under all the smaller frictional irregularities, and any tendency of the lever to remain against either stop is prevented by altering the tension of the strap, and making a corresponding addition to or subtraction from the weight W_1, so as to bring the lever again horizontal.

The word **brake** is used somewhat loosely, not only as referring to the work-measuring dynamometers just described,

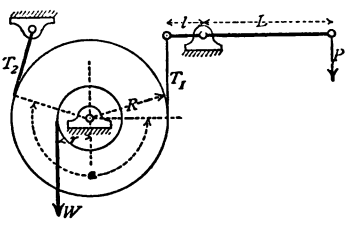


FIG. 368.

but also for a large class of apparatus where work is absorbed in friction without being measured. The theory of the ordinary friction brake of this kind is that given at the commencement of the last section. A general case is shown in Fig. 368, where a weight *W* tends to run down, turning

with it a drum and pulley. It is required to "brake" the weight, that is to stop it from running down, by a pressure P upon a hand lever. The dimensions of the apparatus are given, and the value of W; the magnitude of P has to be found.

$$Wr = (T_2 - T_1)\,R = T_1\,(n - 1)\,R$$

$$W = T_1\,\frac{R}{r}\,(n - 1)$$

$$T_1 l = PL \qquad\qquad T_1 = P\frac{L}{l}$$

$$W = P\frac{L}{l}\cdot\frac{R}{r}\cdot(n - 1)$$

$$P = W\frac{l}{L}\cdot\frac{r}{R}\left(\frac{1}{n - 1}\right)$$

(In these equations n is written for $\epsilon^{f\alpha}$, as on p. 626.) The greater tension is always that *against* which the brake turns; it is therefore always advisable for purposes such as these to

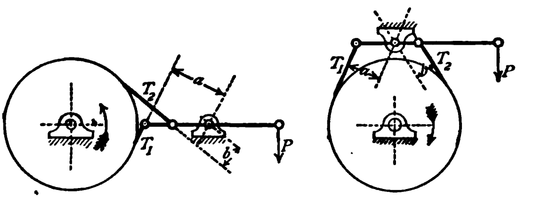


FIG. 369.

connect the brake lever with that end of the strap *towards* which the brake turns, so as to make the pressure P as small as possible.

The friction brake shown in Fig. 369 is called a **differential brake.** The two ends of the brake strap are attached to different points in the same hand lever. If $T_1 a = T_2 b$, the strap is in equilibrium under given conditions

as to friction. To obtain this we must make $\frac{a}{b} = \frac{T_2}{T_1} = \epsilon^{fa}$. If this were done, $P = 0$, so that the smallest possible pressure at P would brake any load. We do not know f accurately enough to carry out these conditions by any means exactly, but even without this a very powerful and handy brake can be made of this type.[1]

§ 80.—PULLEY TACKLE.[2]

THE efficiency of transmission by pulley tackle depends not only on the pin friction of the sheaves, but still more upon the work expended against the stiffness of the rope in bending it round the sheaves and then straightening it again (see p. 629). Physically this is equivalent to an increase of the radius of the resistance (on the bending-on side) and a decrease of the radius of the effort (on the bending-off side), as shown in the sketch, Fig. 370. For a pin link chain the value of the small displacement d can be easily calculated, and from it the loss of efficiency can be found. This is, however, not possible with a rope; the necessary data for the calculation do not exist, and the loss of efficiency must be estimated by use of an empirical formula based on experience. According to Redtenbacher, if F be the pull in a rope of diameter d running on a sheave of radius R, the resistance to bending on or off the pulley is approximately $\left(\frac{1}{3} . \frac{d^2}{R}\right)F$. The formula of Eytelwein, used by Rankine, gives a larger

[1] Descriptions of a number of important brakes will be found in a paper by Mr. W. E. Rich in the *Proc. Inst. Mech. Eng.* July 1876. See also *Zeitschrift d. V. Deutsch. Ing.* 1881, p. 321; *Proc. Inst. Mech. Eng.* July 1858 and July 1877, the latter containing description of Mr. Froude's turbine dynamometer.

[2] The subjects dealt with in §§ 78 to 80 have been perhaps nowhere better handled than by Professor Ritter in his *Technische Mechanik*, to which we are especially indebted in the present section.

resistance $\left(\frac{1}{2\cdot 1}\cdot\frac{d^2}{R}\right)F$. We have no means of knowing on what experiments either is based; the former may perhaps be rightly preferred.

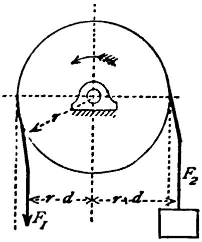


FIG. 370.

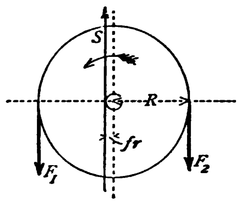


FIG. 371.

Let Fig. 371 represent a sheave of any pulley block, turning in the direction of the arrow, F_1 and F_2 the effort and resistance, that is the tensions in its two cords or "parts." R is the radius of the sheave, r of the pin, and fr that of the friction circle. Without friction (and neglecting also the resistance due to the stiffness of the rope) the force S balancing F_1 and F_2, which is equal in magnitude to their sum, but reversed in sense, would pass through the centre of the sheave, and $F_1 = F_2$. Allowing for friction we know that S must lie at a distance $= fr$ from the centre. Assuming F_1 and F_2 to be parallel, S must of course be parallel to both of them. The magnitudes of F_1 and F_2 must be inversely as their distances from S, *i.e.*,

$$\frac{F_1}{F_2} = \frac{R + fr}{R - fr} = \frac{1 + \frac{fr}{R}}{1 - \frac{fr}{R}}.$$

As $\frac{fr}{R}$ is quite small, we may say very approximately,

$$\frac{F_1}{F_2} = 1 + 2\frac{fr}{R}.$$

Replacing the radii r and R of the pin and sheave respectively, by their diameters d_1 and D, as it is perhaps more convenient to do, we have

$$F_1 = F_2 + 2f\frac{d_1}{D}F_2.$$

We have thus obtained algebraical values, in terms of the tension in a cord, of the two principal added resistances to its motion, and may say in all

$$F_1 = F_2 + 2f\frac{d_1}{D}F_2 + \frac{d^2}{1{\cdot}5D}F_2;$$

or the counter-efficiency of the sheave, which we may call c,

$$= \frac{F_1}{F_2} = 1 + 2f\frac{d_1}{D} + \frac{d^2}{1{\cdot}5D};$$

$$\text{say } F_1 = cF_2.$$

If the pins are well lubricated, so that f is small, it is obvious that the loss on account of the stiffness of the rope will be much greater than that due to pin friction.

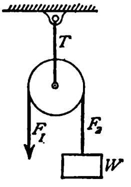


FIG. 372.

Where a cord is simply carried over a pulley, and used to raise a weight, as in Fig. 372, we have for the necessary pull

$$F_1 = cF_2 = cW,$$

and the tension in the suspending link

$$T = F_1 + F_2 = F_1\left(1 + \frac{1}{c}\right)$$

instead of $2F_1$.

In ordinary pulley tackle, as Figs. 373 and 374, assuming all the parts of the rope to be parallel, and all the sheaves

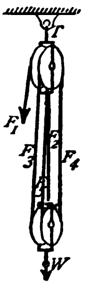


FIG. 373.

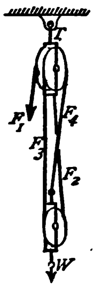


FIG. 374.

of the same diameter, and assuming also that the friction-factor is unaffected by the velocity of the sheaves, we have

$$F_2 = \frac{F_1}{c};$$

$$F_3 = \frac{F_2}{c} = \frac{F_1}{c^2};$$

$$F_4 = \frac{F_3}{c} = \frac{F_1}{c^3}; \text{ and so on.}$$

The weight W is equal to $F_2 + F_3 + F_4 + F_5$, which without friction would be simply $= 4F_1$. Taking friction into account it is

$$W = \frac{F_1}{c} + \frac{F_1}{c^2} + \frac{F_1}{c^3} + \frac{F_1}{c^4}$$

$$= F_1\left(\frac{1}{c} + \frac{1}{c^2} + \frac{1}{c^3} + \frac{1}{c^4}\right)$$

$$= \frac{F_1}{c^4}(c^3 + c^2 + c + 1)$$

$$= F_1\left(\frac{c^4 - 1}{c^5 - c^4}\right)$$

Or in general, if n be the whole number of sheaves in the two blocks of the tackle (or what is the same thing, the number of plies of rope whose tensions together balance the weight W, namely 4 in Fig. 373 and 3 in Fig. 374), the weight which can be lifted by any given pull F_1 is

$$W = F_1 \left(\frac{c^n - 1}{c^{n+1} - c^n} \right).$$

Without friction the tackle would lift a weight

$$W = nF_1.$$

The counter-efficiency of the tackle as a whole is therefore

$$n \left(\frac{c^{n+1} - c^n}{c^n - 1} \right).$$

A probable enough value of c under ordinary conditions may be 1·1, and with this value the counter-efficiency of a tackle of five sheaves is 1·31.

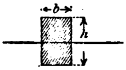
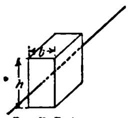
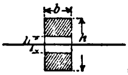
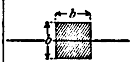
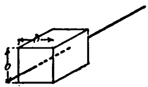
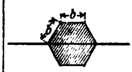
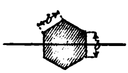
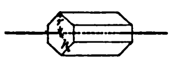
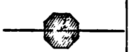
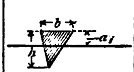
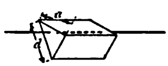
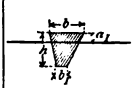
Figure	Moment of Inertia about axis through Mass centres	Solid	Moment of Inertia about axis through Mass centres
			M= Mass of Solid
 *Rectangle*	$\frac{bh^3}{12}$	 *Rect^r Prism*	$\frac{M(b^2+h^2)}{12}$
	$\frac{b[h^3-h_1^3]}{12}$		
 *Square*	$\frac{b^4}{12}$	 *Square Prism*	$\frac{M b^2}{6}$
	$\frac{b^4}{12}$		
 *Hexagon*	$\frac{5\sqrt{3}\,b^4}{16}$		
	$\frac{5\sqrt{3}\,b^4}{16}$	 *Prism with section any regular polygon*	$\frac{M}{3}\left[\frac{r^2}{2}+k^2\right]$
 *Octagon*	$\frac{1+2\sqrt{2}\,r^4}{6}$		
 *Triangle*	$\frac{bh^3}{36}$ $\left(a_1+\frac{h}{3}\right)$	 *Prism with section a right angled triangle*	$\frac{M d^2}{18}$
	$\frac{b^2+4bb_1+b_1^2}{36(b+b_1)}h^3$ $\left(a_1-\frac{b+2b_1}{b+b_1}\frac{h}{3}\right)$		

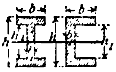
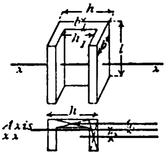
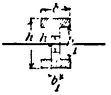
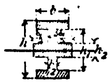
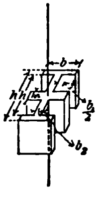
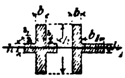
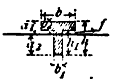
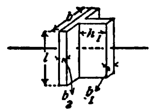
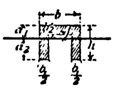
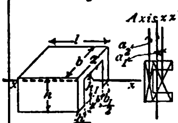
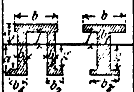
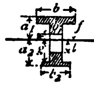
Figure	Moment of Inertia about axis through Mass centres	Solid	Moment of Inertia about axis through Mass centres M = Mass of Solid
	$\frac{bh^3-[b-b_1]h_1^3}{12}$	 	$l\left[b[h-h_1]\left[\frac{h^2+l^2}{12}+a_2^2\right]\right.$ $\left.+b_1h_1\left[\frac{b_1^2+l^2}{12}+a_1^2\right]\right]$
	$\frac{b\left[h^3-h_1^3\right]+b_1\left[h_1^3-h_2^3\right]}{12}$		
	$\frac{bh^3-(b-b_2)h_1^3+b_1h_2^3}{12}$		$\frac{bh^3-[b-b_2]h_1^3+b_1h_2^3+b^3[h-h_1]}{12}$ $\frac{+[b_1+b_2]^3h_2+b_2^3[h_1-h_2]}{}$
	$\frac{bh^3+b_1h_1^3}{12}$		
	$\frac{1}{3}\left[b\left(a_1^3-f^3\right)+b_1\left[f^3+a_2^3\right]\right]$		$b_1h_1l\left[\frac{b_1^2+l^2}{12}\right]+blb_2\left[\frac{b^2+l^2}{12}\right]$
	$\left(a_1=\frac{bb_2^2+b_1h_1[h+b_2]}{2\left[bh-[b-b_1]h_1\right]}\right)$	 	$l\left[h_1\left[b\left[\frac{h_1^2+b^2}{12}\right]+b_3\left[\frac{h_1^2+b_3^2}{12}\right]+b_1a_1^2\right]\right.$ $\left.+bb_2\left[\frac{b_2^2+b^2}{12}+a_2^2\right]\right]$
	$\frac{1}{3}\left[b\left(a_1^3-f^3\right)+b_1\left(f^3+g^3\right)+b_2\left(a_2^3-g^3\right)\right]$ (a_1 to be found graphically)		
	$\frac{1}{3}\left[b\left[a_1^3-f^3\right]+b_1\left[f^3+g^3-k^3-i^3\right]+b_2\left[a_2^3-g^3\right]\right]$ (a to be found graphically)		

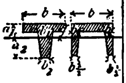
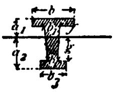
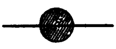
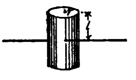
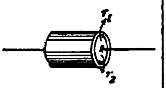
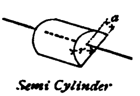
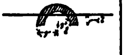
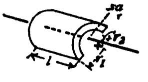
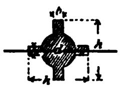
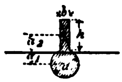
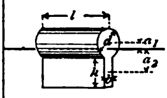
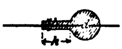
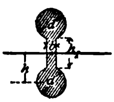
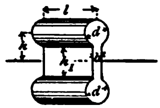
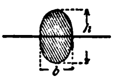
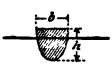
Figure	Moment of Inertia about axis through Mass centres	Solid	Moment of Inertia about axis through Mass centres
			M = Mass of Solid
	$\frac{1}{3}\left[\frac{b_1-b_2}{4\left[f+a_2\right]}\left[a_2^4+3f^4+4a_2f^3\right]+b\left[a_1^3-f^3\right]+b_2\left[f^3+a_2^3\right]\right]$		
	$\frac{1}{3}\left[\frac{b_1-b_2}{4(f+g)}\left[g^4+3f^4+4gf^3\right]+b\left[a_1^3 f^3\right]+b_2\left[f^3+g^3\right]+b_3\left[a_2^3-g^3\right]\right]$ (a_1 *to be found graphically*)		
		 *Sphere*	$M\frac{2}{5}r^2$
 *Circle*	$\frac{\pi}{64}d^4$	 *Cylinder*	$M\frac{r^2}{2}$
			$M\left[\frac{r^2}{4}+\frac{l^2}{3}\right]$
	$\frac{\pi}{64}\left[d^4\ d_1^4\right]$		$\frac{M}{2}\left[r_1^2+r_2^2\right]$
 *Semi Circle*	$r^4\left[\frac{\pi}{8}-\frac{8}{9\pi}\right]=0{\cdot}110r^4$ $\left[a=\frac{4r}{3\pi}=0.4244r\right]$	 *Semi Cylinder*	$M\left[\frac{r^2}{2}-a^2\right]$
	$\frac{\pi}{8}\left[r_1^2\ r_2^2\right]\times\left[r_1^2+r_2^2-4a^2\right]$ $a=\frac{4}{3\pi}\left[\frac{r_1^2+r_1r_2+r_2^2}{r_1+r_2}\right]$		$M\left[\frac{r_1^2+r_2^2}{2}-a^2\right]$

Figure	Moment of Inertia about axis through Mass centres	Solid	Moment of Inertia about axis through Mass centres
			M = Mass of Solid
	$\frac{1}{12}\left[\frac{3\pi}{16}d^4+b\left[h^3-d^3\right]+b^3\left[h-d\right]\right]$		
	$\frac{\pi d^2}{4}\left[\frac{d^2}{16}+a_1^2\right]+bh\left[\frac{h^2}{12}+a_2^2\right]$ $\left(a_1=\frac{bh\left[\frac{h}{2}+d\right]+\frac{\pi d^3}{8}}{bh+\frac{\pi d^2}{4}}\right)$		$l\left[\frac{\pi d^2}{4}\left[\frac{d^2}{8}+a_1^2\right]+bh\left[\frac{h^2+b^2}{12}+a_2^2\right]\right]$
	$\frac{\pi d^4}{64}+\frac{hb^3}{12}$		
	$\pi d^2\left[\frac{d^2}{32}+\frac{h^2}{2}\right]+\frac{bh_1^3}{12}$		$l\left[\frac{\pi d^2}{2}\left[\frac{d^2}{8}+h^2\right]+bh_1\left[\frac{h_1^2+b^2}{12}\right]\right]$
	$\frac{\pi d^4}{32}+\frac{h_1b^3}{12}$		
 *Ellipse*	$\frac{\pi}{64}bh^3$	 *Ellipsoid*  *Elliptical Cylinder semi axes a & b*	$M\frac{2b^2}{5}$ $M\frac{a^2+b^2}{4}$
 *Parabola*	$\frac{8}{175}bh^3$	 *Paraboloid*	$M\frac{a^2}{3}$

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RICHARD CLAY AND SONS, LONDON AND BUNGAY.